Linear response: Ion plasma waves

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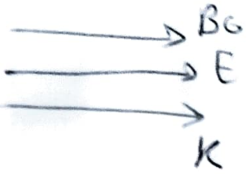
## (i) Electrostatic waves: $(K \parallel E)$ & $\left(\frac{\partial B_1}{\partial t}=0\right)$:

$(B_0=0)$

### (1) unmagnetized OR Parallel wave $(K \parallel B_0)$:

$$w^2 = \frac{V_s^2 K^2}{1+\lambda_{Di}^2 K^2} + V_{thi}^2 K^2$$

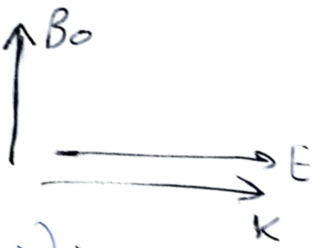


- Ion aCoustic wave     Ion thermal wave
- Longitudinal wave     • Compression/refraction in density.

### (2) Perpenduclar wave $(K \perp B_0)$:

- Ion cycltron waves / magnetized ion aCoustic wave

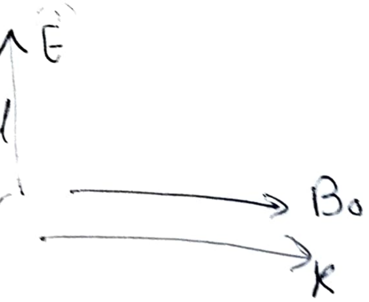
$$w^2 = \Omega_c^2 + \frac{V_s^2 K^2}{1+\lambda_{Di}^2 K^2} + V_{thi}^2 K^2$$



## (ii) Electromagnetic waves $(K \perp E$ & $\frac{\partial B_1}{\partial t} \neq 0)$:

### (1) Parallel wave $(K \parallel B_0)$:

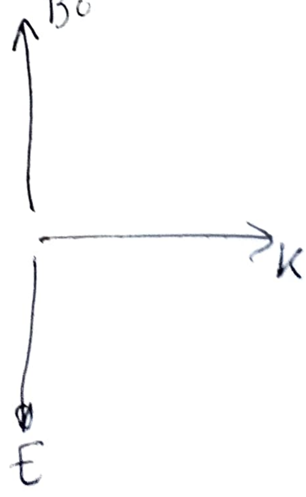
$$w^2 = V_A^2 K^2 ;$$ Alfven wave → torsional / → shear



- transverse wave
- magnetic tension $T_B = \frac{B_0^2}{\mu_0}$ ; $V_A = \frac{B_0^2}{\mu_0 \rho}$

### (2) Perpendcular wave $(K \perp B_0)$:

$$w^2 = (V_A^2 + V_S^2) \pm \sqrt{(V_A^2 + V_c^2) - 4 V_A^2 V_S^2}$$



- Longitudinal wave
- magnetic and thermal pressure.

# Electrostatic wave:

- the condition is

$$\vec{K} \parallel \vec{E} \rightarrow \nabla \times E = 0, \quad \frac{\partial B_1}{\partial t} = 0$$

- there is no perturbation in $\tilde{B}$.
- the system of equations:

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot v = 0$$

$$\frac{\partial \tilde{P}}{\partial t} = V_s^2 \frac{\partial \rho_0}{\partial t} = - V_s^2 \rho_0 \vec{\nabla} \cdot \vec{V}.$$

$$\rho_0 \frac{\partial v}{\partial t} = q(E + V \times B_0) - \nabla P.$$

$$\nabla \cdot E = n_i - n_e.$$

$$n_e \sim e^{\phi}.$$

د. [illegible]

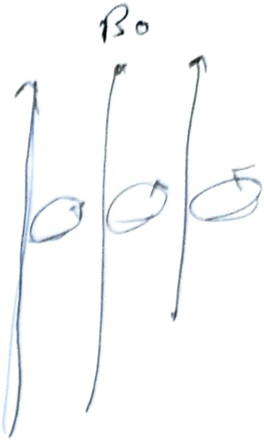
Densities

# ① Ion thermal wave:

- $\omega = V_{th} K$
- $\frac{\partial^2 \rho}{\partial t^2} - V_{th}^2 \nabla^2 \rho = 0 \rightarrow \frac{\partial P}{\partial t} = --$
- longitudinal wave: $\tilde{V}_i \parallel K$
- Parallel wave. $K \parallel B_0$
- speed: $V_{th} = \frac{KT_i}{m_i}$.
- Compr mechanism: - restoring force = $KT_i$
  - Inertia = $m_i$
- Compression in
  - → mass density
  - → velocity field.
  - → pressure.

## n cycltron wave / magnetized IAW:

(i) $$\frac{\partial^2 \rho}{\partial t^2} - V_s^2 \nabla^2 \rho - \omega_c^2 \rho = 0$$



$$\omega^2 = \omega_c^2 + k^2 V_s^2$$

- Perpendicular wave $k \perp B_0$.
- longitudinal wave.
- there is oscillation first with $\omega_c$

  then a ~~wave~~ IAW

## (② Ion acoustic wave:

- $$\frac{\partial^2 \rho}{\partial t^2} - V_s^2 \nabla^2 \rho = 0$$
- $$\omega^2 = V_s^2 k^2 .$$
- longitudinal wave.
- parallel wave.
- $$V_s = \sqrt{\frac{KT_e}{m_i}}$$
- mechanism; - restoring force = $KT_e$.
  - Inertia = $m_i$ .
- Compression in → mass density
  → velocity
  → pressure.

# Electromagnetic waves:

- the condition of EMWs: $B = B_0 + \tilde{B}$

$K \perp E \rightarrow \nabla\times E \neq 0 \therefore \frac{\partial \tilde{B}}{\partial t} \neq 0$

→ there is a perturbation in $\tilde{B}$ → direction → shear, twist, bend; → magnitude → compression, refraction.

- The system of equations:

$$\boxed{\frac{\partial \rho}{\partial t} = -\rho_0\, \vec{\nabla}\cdot\vec{v}}$$

Compressibility

$$\boxed{\frac{\partial P}{\partial t} = c_s^2 \frac{\partial \rho_0}{\partial t} = -c_s^2 \rho_0\, \vec{\nabla}\cdot\vec{v}}$$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}_0) = -\vec{B}_0(\vec{\nabla}\cdot\vec{v}) + (\vec{B}_0\cdot\vec{\nabla})\vec{v}}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\nabla P + \frac{1}{\mu_0}(\nabla\times B)\times B$$

$$= -\nabla P - \nabla \frac{B_0\cdot\vec{\tilde{B}}}{\mu_0} + \frac{1}{\mu_0}(\vec{\tilde{B}}_0\cdot\vec{\nabla})\vec{B}$$

$$\boxed{\rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla\Big(\underbrace{P}_{\text{thermal pressure}} + \underbrace{\frac{B_0\cdot B}{\mu_0}}_{\text{magnetic pressure}}\Big) + \frac{1}{\mu_0}\underbrace{(B_0\cdot\nabla)B}_{\text{magnetic tension or shear/straining along } \vec{B}}}$$

①

## * The plasma response to the perturbation:

$$\left(\omega^2 - k V_A^2\right)\left[\omega^4 - \left(V_A^2 + V_s^2\right)k^2\omega^2 + k^4 V_A^2 V_s^2\right] = 0$$

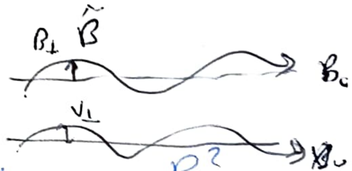
### ① shear / torisonal Alfrèn wave

- the governing eq is:

$$\left.\begin{aligned} &\frac{\partial^2 B_\perp}{\partial t^2} - V_A^2 \nabla_\parallel^2 B_\perp = 0 \\ &\frac{\partial^2 V_\perp}{\partial t^2} - V_A^2 \nabla_\parallel^2 V_\perp = 0 \end{aligned}\right\} \quad \frac{\partial \tilde{B}_\perp}{\partial t} \sim (B_0 \cdot \nabla)\tilde{V}_\perp . \quad \boxed{\omega = V_A k} \sim 10^5/s .$$

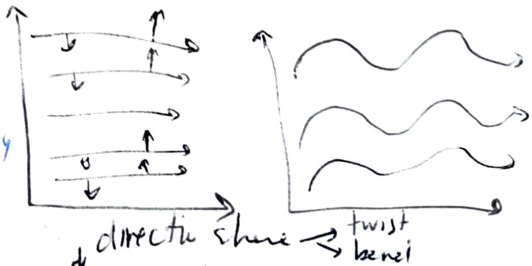
- Parallel wave: $k \parallel B_0$
- transverse wave: The perturbation in $(B_\perp \;\&\; V_\perp) \perp k$.
- the propagation speed is

Alfrèn speed $= \frac{B_0^2}{\mu_0 \rho_0}$



- mechanism: - restoring force = magnetic tension $= \frac{B_0^2}{\mu_0}$

  - Ineria = Ion density $= \rho_0$.



- properties:
  - $\vec{\nabla} \cdot \vec{V} = 0$, there is no change in magnetic field density
    ↳ no compression.
  - $\frac{\partial B_\perp}{\partial t} \sim (\tilde{B}_0 \cdot \tilde{\nabla}) V_\perp$ → there is shear along $\underline{B}$ only.
- Analogy: wave in string. $V^2 = \cancel{\sqrt{}}\,\frac{T}{\mu} = \frac{B_0^2}{\mu_0 \rho_0}$.

Line Solution of the second bracket:

$$\omega^2 = K^2 V_\pm^2 \;;$$

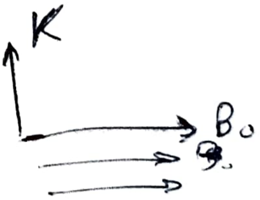
$$V_\pm = \frac{1}{2}\left[V_A^2 + V_c^2 \pm \sqrt{(V_A^2 + V_s^2)^2 - 4V_A^2 V_s^2}\right]$$

② fast magnetosonic wave / Compressional Alfven wave

- $\omega^2 = V_+^2 K^2 \simeq K^2 (V_A^2 + V_s^2) \sim 10^6/s$
- the equation:

$$\left[\frac{\partial^2}{\partial t^2} - (V_A^2 + C_s^2)\nabla^2\right]\left[P + \frac{B_\parallel B_0}{\mu_0}\right] = 0$$

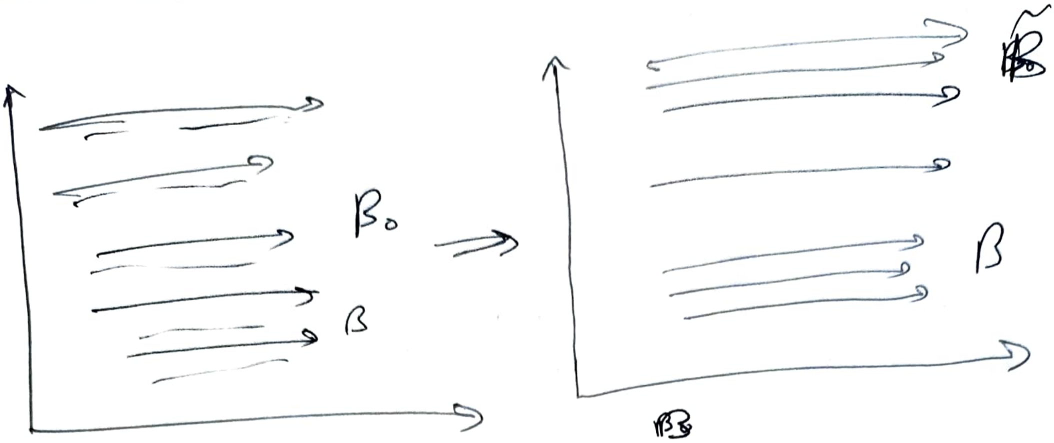
- Longitudinal wave. $\vec{B}_\parallel \parallel \vec{K}$ : Perturbation $\parallel$ Propagation.
- Perpendicular wave: $K \perp B_0$
- Propagation speed is $V = V_A^2 + V_s^2$



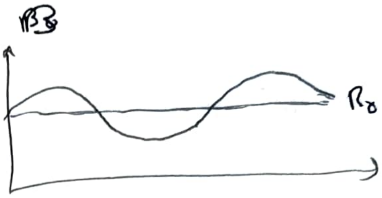
$$V_A^2 = \frac{B_0}{2\mu_0 \rho_0} \;,\; V_s^2 = \frac{P_0}{\rho_0}.$$

- mechanism: restoring force → magnetic pressure $\frac{B_0^2}{2\mu_0}$
  → thermal pressure $P_0$.

* Properities:



$K \uparrow$ ties



- $\tilde{\nabla} \cdot \tilde{V} \neq 0 < 0$ (Converse)

there is a compression.

- $\frac{\partial \rho}{\partial t} > 0$ : there is Compressi in mass density.
- $\frac{\partial P}{\partial t} > 0$ : " " " " pressure.
- $\frac{\partial \tilde{B}}{\partial t} > 0$ : " " " " magnetic field.

## (i) Slow magnetosonic / magnetoacoustic wave:

- In

$\omega^2 = k^2 V_-^2 \simeq$ ~~$V_s^2 - V_A^2 k_{\parallel}^2$~~ $\simeq C_s^2$

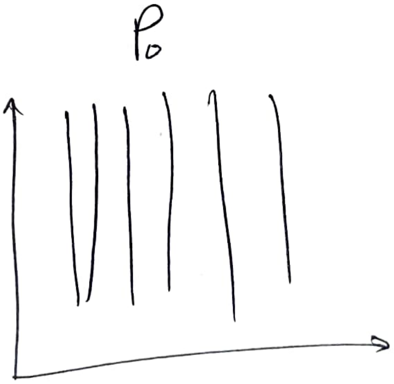
- the equation: $\simeq V_s^2 - V_A^2 \simeq V_s^2 k_{\parallel}^2 \sim 10^4/s$.

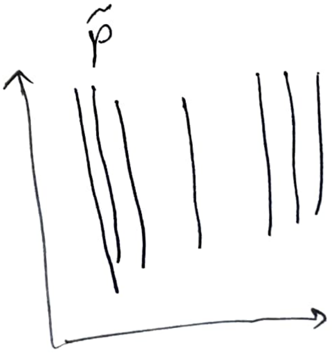
$$\frac{\partial \tilde{p}}{\partial t^2} - C_s^2 \nabla_{\parallel}^2 \tilde{p} = 0 .$$

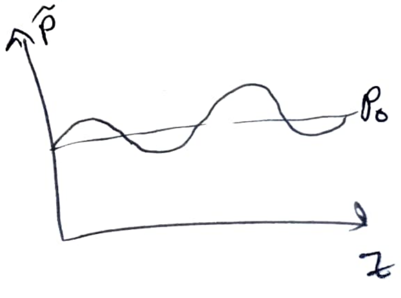
$$\frac{\partial \tilde{p}}{\partial t} = - C_s^2 \rho_0 \bar{\nabla} \cdot \tilde{v} .$$

- longitudinal wave.
- parallel wave. $\bar{k} \parallel \bar{B}_0$.
- propagation speed: $V_s^2 = \frac{P_0}{\rho_0}$.
- mechanism: - restoring force = thermal pressure $P_0$.
  - Inertia = ions mass density $\sim \rho_0$.

- properties:
- It just sound wave propagate freely along the magnetic field $\bar{B}$:







* the distinction betwe fast and slow:

$$\frac{B_0 \cdot B}{\mu_0} = \frac{V_A^2}{V_s^2}\left(1 - \frac{V_s^2}{V^2}\right) p$$

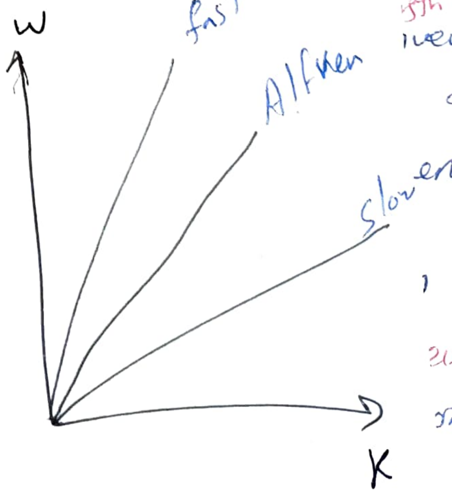
magnetic Pressure = ( Sign ) thermal Pressure.

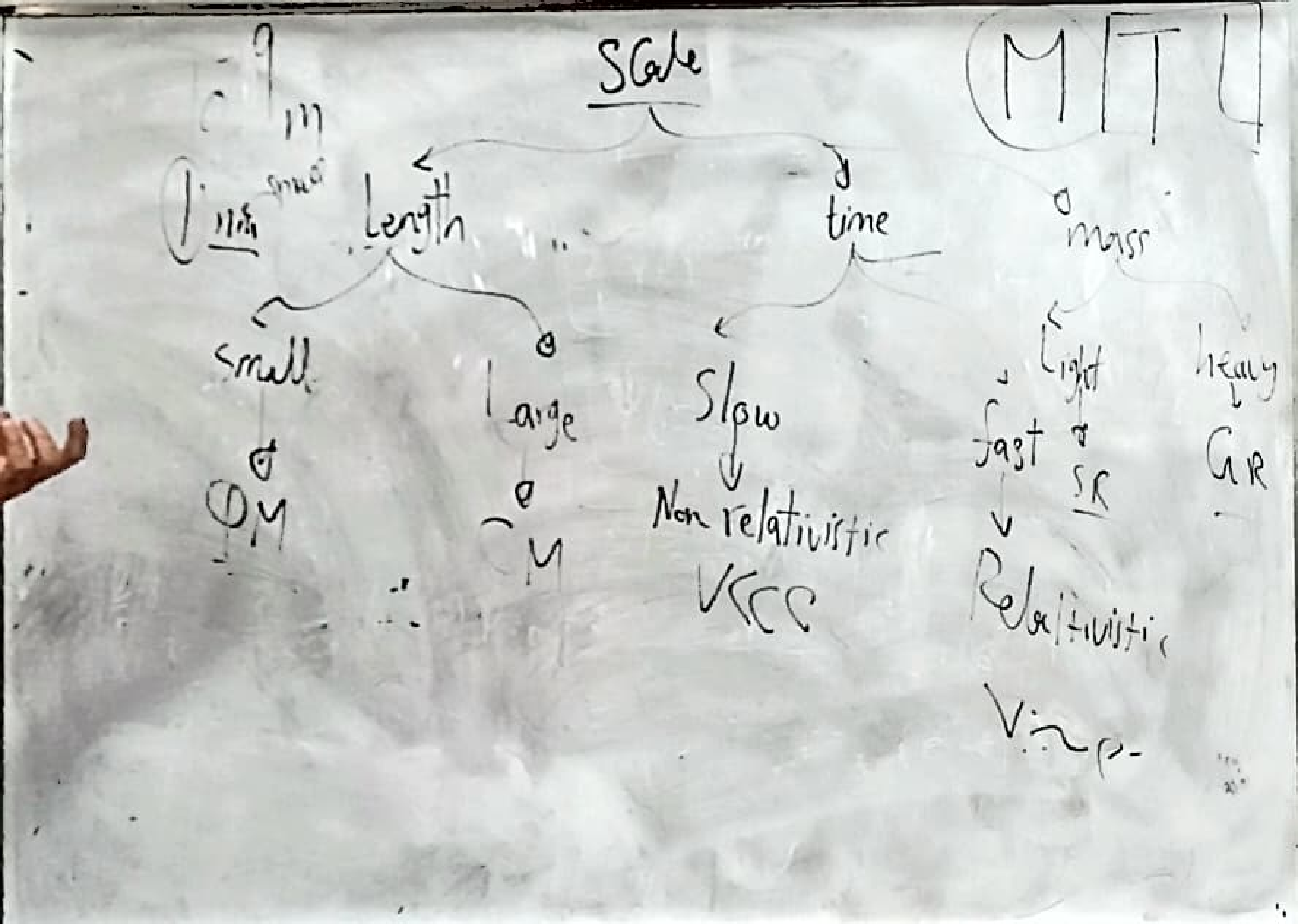
(i) if $V = V_+ > C_s$:

- $p$ and $B_0 \cdot B$ has the Same Sign
- reinforce each other.
- Constructive interference.

(ii) if $V = V_- < C_s$:

- $p$ and $B_0 \cdot B$ has opposite direction
- Weaken each other.
- Destructive interference.




Scale
M | T | L
small
Length
time
mass
small
Large
Slow
fast
Light
heavy
QM
CM
Non relativistic
V<<c
SR
GR
Relativistic
V≈c

# Plasma system:

| Grandi | Parents | Grandsons |
|---|---|---|
| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| Dust | Ions | electrons. |

$12\ m_i$

$\tau$: plasma response time

## * time scale:

(1) $\frac{\omega_{pi}}{\omega_{pD}} \sim \frac{M_d}{m_i} \sim 10^{12}\ \tau_i$

$\tau_i \sim 1\,\text{sec}$

(2) $\frac{\omega_{pe}}{\omega_{pi}} \sim \frac{m_i}{m_e} \sim 2000 \sim 10^{3}\ \tau_e$

## Plasma system:

| | الأبناء | الآباء | الأجداد |
|---|---|---|---|
| Species | electron frame | Ion frame | Dust frame |
| | fast time scale | slow time scale | very slow time scale |
| e | e → fluid Eqs<br>$\frac{dv_e}{dt} \neq 0$ | $V_e = C$<br>Distribution | $V_e = C$<br>Distribution |
| i | $\frac{dv_i}{dt} = 0$<br>Stationary/fixed | $\frac{dv_i}{dt} \neq 0$<br>fluid Eqs | $V_i = C$<br>Distribution |
| D | $\frac{dv_D}{dt} = 0$<br>Stationary/fixed | $V_D = 0$<br>fixed | $\frac{dv_D}{dt} \neq 0$<br>fluid Eqs |

## Plasma system:

| Species | الأبناء<br>electron/-frame ($\sim ps$)<br>fast time scale | الآباء<br>Ion frame ($\sim \mu s$)<br>Slow time scale | الأجداد<br>Dust frame ($\simeq ms$)<br>very slow time scale |
|---|---|---|---|
| e/p | e → Fluid Eqs<br>$\frac{dv_e}{dt} \neq 0$ | $V_e = C$<br>, Distribution. | $V_e = C$<br>Distribution |
| i | $\frac{dv_i}{dt} = 0$<br>Stationary/fixed | $\frac{dv_i}{dt} \neq 0$<br>Fluid eqs | $V_i = C$<br>Distribution |
| D | $\frac{dv_D}{dt} = 0$<br>Station/fixed | $V_D = 0$<br>fixed. | $\frac{dv_D}{dt} \neq 0$<br>Fluid Eqs. |

# Ion Plasma Waves

$\uparrow B \sim \overset{\rightarrow\, 5nT}{B_0} + \tilde{B} \rightarrow$ Perturbed (Internal)

(external) ambient

Electrostatic

(i) $\frac{\partial \vec{B}}{\partial t} = 0$ , $\vec{\tilde{B}} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(ii) $\vec{\nabla} \times \vec{E} = 0$

(iii) $\vec{\nabla} \times \vec{\nabla} \phi = 0$

(iv) $\vec{E} = -\nabla \phi$

$\phi$ : electrostatic Potential

Electromagnetic

$\frac{\partial B}{\partial t} \neq 0$ , $\vec{\tilde{B}} \neq 0$

$$\uparrow \; B \sim \overset{5nT}{B_0} + \tilde{B}$$

$B_0$: (external) ambient

$\tilde{B}$ → Perturbed (Internal)

# Ion Plasma Waves

## Electrostatic

(i) $\frac{\partial \vec{B}}{\partial t} = 0$ , $\vec{\tilde{B}} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(ii) $\vec{\nabla} \times \vec{E} = 0$

(iii) $\vec{\nabla} \times (\vec{\nabla} \phi) = 0$

(iv) $\vec{E} = -\vec{\nabla} \phi$

$\phi$ : electrostatic Potential

## Electromagnetic

(i) $\frac{\partial B}{\partial t} \neq 0$ , $\vec{\tilde{B}} \neq 0$

(ii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

# Linear Ion Plasma Waves

Linear effect ←

Non linear effect

$$f = \underset{1000}{f_0} + \underbrace{\overset{\text{equilibrium}\ (t=0)}{\tilde{f}^1}}_{\text{small}} + \tilde{f}^2 + \tilde{f}^3 + \dots + \tilde{f}^{10}$$

Perturbed

$\tilde{f} \sim 0.1$

$\tilde{f}^2 \sim 0.01$

$\tilde{f}^3 \sim 0.001$

$\tilde{f}^1 \sim 10$

$\tilde{f}^2 \sim 100$

$\tilde{f}^3 \sim 1000$

# Linear Ion Plasma Waves

x Electrostatic waves

## ① Ion thermal wave

$$\frac{1}{2} m_i v^2 = \frac{3}{2} k T_i$$

$$V_{th} = \sqrt{\frac{\boxed{k T_i}}{\boxed{m_i}}}$$

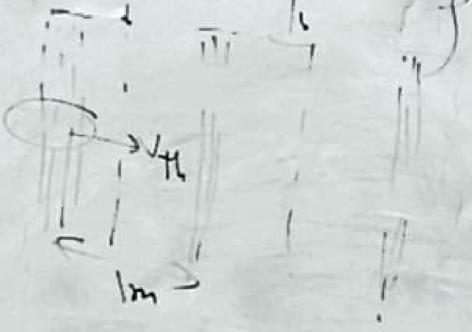
← Thermal restoring force

← Inertia

① Longitudinal

② Compression & rarf.

$$V = \frac{R}{I_i} = \frac{P}{\rho}$$

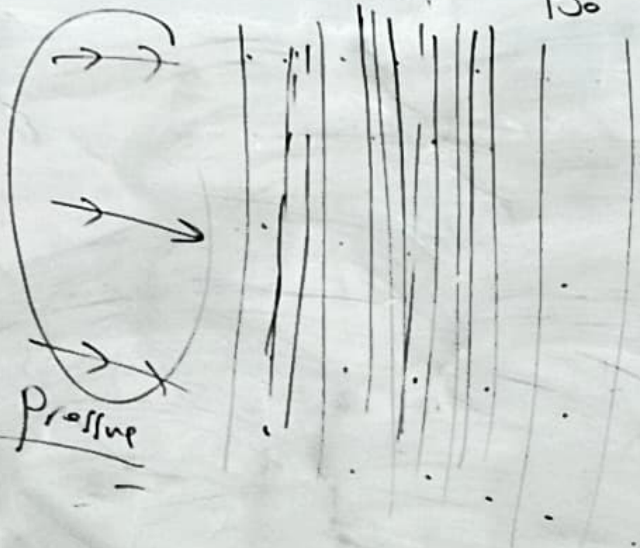


# Linear Ion Plasma Waves 1T $B_o$

x Electromagnetic wave

## Ion Magnetosonic Wave.

$$V = V_S + V_A$$



magnetic pressure

# Linear Ion Plasma Waves II

x Electromagnetic wave:

**Ion Alfvén wave:**

- transverse wave:
- $V_A = \frac{R}{I} = \frac{B_0^2/\mu}{m_i}$

$B = B_0 \pm \tilde{B}$

$V = \sqrt{\frac{T}{\mu}}$

$\frac{B_0^2}{\mu} = T$

↳ magnetic tension.