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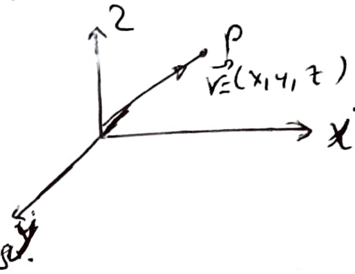
Kinetic model
(statistical mechanics & mesoscopic)

*** Phase space:**

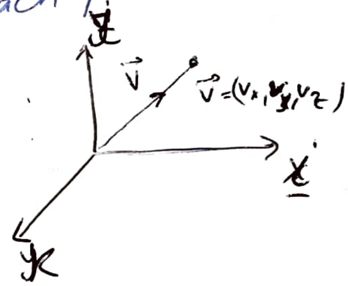
i- Configuration space: At any instant of time, each particle in the plasma can be localized by a position vector ($\vec{r}$) drawn from the origin of a coordinate system to the center of mass of the particle. In Cartesian coordinates:



$$\vec{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$$

- No of dimensions: $3N$; N: no of particles.

ii- Velocity space: At any instant of time, each particle can be localized by a velocity vector ($\vec{v}$):



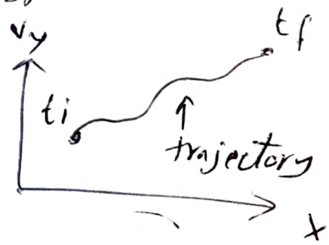
$$\vec{v} = V_x\,\underline{i} + V_y\,\underline{j} + V_z\,\underline{k}.$$

- No of dimensions: $3N$; N: no of particles.

3- phase space:

- The instantanous dynamic state of each particle can be specified by its position and velocity vector, so ~~the phase~~

Phase space = Configuration space + velocity space



- For each particle you need = 6D.
- For N particles = 6ND.
- No of dimensions: 6ND.
- Each ~~particle~~ system represents as a point, when the system moves, it forms a trajectory in phase space

Note that:

(i) μ space: 6D & each particle represent a point (for $N \to N$ points.

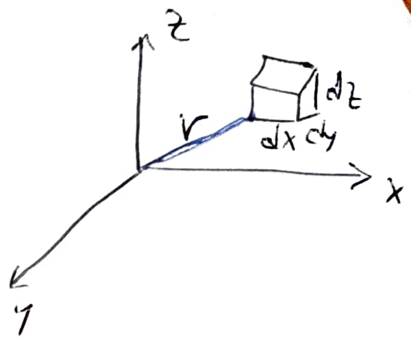
(ii) Γ space: 6ND & each system " " (for $N \to 1$ point.

* Differential elements:

~~The confi~~

1- Configuration volume element:

- A small element of volume in configuration space is represented by $\boxed{d^3r = dx\,dy\,dz}$.



- This differential element d^3r is
  - a mathematically infinitesimal quantity.
  - finite element, sufficiently large to contain a very large number of particles.
  - sufficiently small in comparison with the spatial variation of physical quantity, i.e. the physical doesn't change inside the element.
  - Exp: In gas containing 10^{18}, if we take $d^3r = 10^{-12}\,m^{-3}$ there are 10^6 molecules inside d^3r → 10^{12} ensamble point

- A particle at position $\vec{r}$ inside d^3r, around a terminal point, means

  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

  the x coordinate of the particle lies between x, $x + dx$.
  " y " " " " " " y, $y + dy$.
  " z " " " " " " z, $z + dz$.

  no of particles inside in V: $n = 10^6\ \frac{\#}{m^3}$

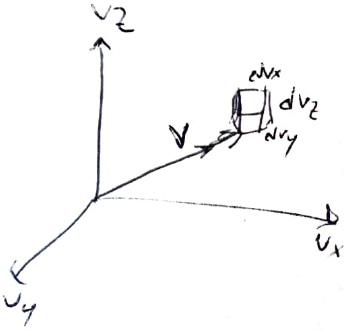
  $N = 10^{18}\ \#$

  the particle has arbitrary velocity in velocity space.

  no of ensambles $= 10^{12}$

2- Velocity volume element:

- A small element of volume in velocity space is represented by $d^3v = dv_x\,dv_y\,dv_z$.



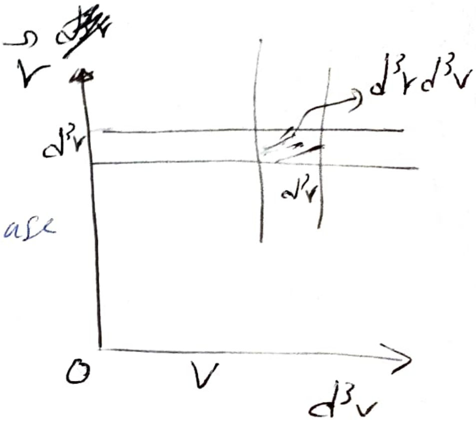
- A particle inside d^3v, around the terminal point of the velocity vector $\vec{v}$.
  - its v_x velocity lies between v_x and $v_x + dv_x$
  - " v_y " " " v_y and $v_y + dv_y$
  - " v_z " " " v_z " $v_z + dv_z$.

Phase Volume element:

In phase space (μ-space) a differential element of volume is a six-dimensional cube:

$$dV = d^3r\, d^3v = dx\,dy\,dz\; dv_x\, dv_y\, dv_z$$

4- differential number: ~~..~~



- the number of particles inside the volume element $d^3r\, d^3v$ around the phase space coordinates $(\vec{r}, \vec{v})$ at instant t is

$$d^6N(\vec{r}, \vec{v}, t)$$

- the total number is

$$N = \int d^6N\; d^3r\, d^3v.$$

*Distribution function:

- the phase space ~~density~~ distribution function is given by

$$f(\vec{r}, \vec{v}, t) = \frac{d^6N(\vec{r}, \vec{v}, t)}{d^3r\, d^3v} = \frac{\text{no of particles}}{\text{Volume element}}$$

d^3r $\boxed{d^6N}$ d^3v

- where: $f(\vec{r}, \vec{v}, t)$: the number density of particles inside $d^3r\, d^3v$.

- mass density:

$$\rho = \frac{M}{V}$$

$$dm = \rho\, dv.$$

- if the density is not homogenous

$$dm = \rho(x)\, dv.$$

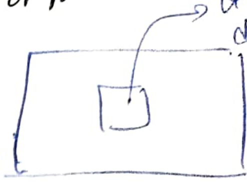
$f(\vec{r}, \vec{v}, t)$	$n(\vec{r}, t)$
• number density function in phase space	• Configuration space number density
• $\frac{\#}{m^3 \cdot v^3} = \frac{N^6}{V^6}$	• $\frac{\#}{m^3}$ $\frac{N^3}{V^3}$
• distributed	• random function.

- Number density:

$$f = \frac{d^6N}{d^3r\,d^3v} = \frac{d^6N}{d\mathcal{V}} \rightarrow \frac{N}{\mathcal{V}}$$

$d\mathcal{V}$: have to be small compared to the overall spatial extension of the points, but still keep it large enough to have a sufficient large number of points inside the volume

- $d^6N = \underbrace{f\ d\mathcal{V}}$ → constant


$\frac{dn}{dv}$

- if the configuration space is not homogeneous

$d^6N = \underbrace{f(x)\ d\mathcal{V}}$ → function of x.

$d\mathcal{V}$: volume of the ensamble in phase space has 6D.

- if velocity space is not isotropic:

$d^6N = \underbrace{f(\vec{r},\vec{v})\ d\mathcal{V}}$ → function of $\vec{r}, \vec{v}$.

$$f(\vec{r},\vec{v},t) = \lim_{\delta \mathcal{V} \to 0^+} \frac{\delta^6 N}{\delta \mathcal{V}}$$

- if the density is not static,

$d^6N = \underbrace{f(\vec{r},\vec{v},t)\ d\mathcal{V}}$ → function of $\vec{r}, \vec{v}, t$.

So, $f(\vec{r},\vec{v},t)$: phase space density function

- if f is distributed in specific way, so $f(\vec{r},\vec{v},t)$ is called: Phase-space number density distribution function

- probability distribution function

total number of particles in phase space

$$N = \iint d^6N = \iint f(\vec{r},\vec{v},t)\ d^3r\,d^3v.$$

$$1 = \iint \frac{f(\vec{r},\vec{v},t)}{N}\ d^3r\,d^3v$$

$$= \iint P(\vec{r},\vec{v},t)\ d^3r\,d^3v$$

$n(\vec{r},t) = \int f(\vec{r},\vec{v},t)\ d^3v$
↳ Configuration space number density function

$P(\vec{r},\vec{v},t)$: Probability distribution function.

$$dP(\vec{r}+d\vec{r}, \vec{v}+d\vec{v}, t) = P(\vec{r},\vec{v},t)\ d^3r\,d^3v.$$

Classification of distribution function:

Equilibrium Vs Nonequlibrium:

If the distribution function f doesn't depend on t is called "Equilibrium distribution function".

- if " " " f depend on t is called "Non equilibrium distribution function".

② Homogenous Vs Non homogenous:

- if the distribution function f doesn't depend on $\vec{r}$, it is called " Homogenous distribution function".
- If the distribution " f depends on $\vec{r}$, it is called". Non homogenous distribution function".

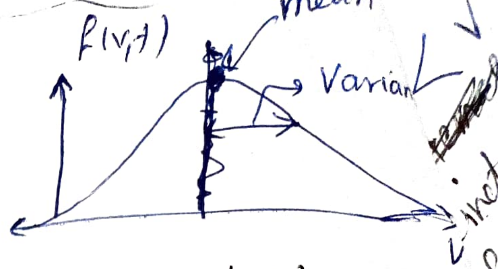
③ Isotropic Vs Anisotropic:

- If the distribution function f depends only on the magnitude of $v = |v|$, it is Called " Isotropic distribution function".
- if " " " " depends on the orientation of $\vec{v}$, it is Called " Anisotropic distribution function".

* Conditions of the distribution function:

① continous ② real positive ③ finite.

* Moments of distribution function: Averge values and macroscopic variables:



(i) Moments:

$$M_n = \int v^n f(x,v,t)\, dv$$

Zero moment $M_0 = \int v^0 f(x,v,t)\, dv$ = Area under (total number)

1st moment $M_1 = \int v^1 f(x,v,t)\, dv$ = Averge ~~or~~ mean value (momentum)

2nd moment $M_2 = \int v^2 f(x,v,t)\, dv$ = Variance (energy)

3rd moment $M_3 = \int v^3 f(x,v,t)\, dv$ = Skeweness (symmetry around mean). (Heat)

Ensamble averges: set of many replicas of the system, which are identical in all other respects apart from being in different states at an instant of time.

(ii) Averge values and macroscopic variables:

- The averge value of the property $\chi(\vec{r},\vec{v},t)$ for the particle

$$\langle \chi(\vec{r},\vec{v},t)\rangle = \frac{1}{n(\vec{r},t)} \int \chi(\vec{r},\vec{v},t)\, f(\vec{r},\vec{v},t)\, d^3v$$

(i) $\chi(\vec{r},\vec{v},t) = v^0$

$$\langle v^0 \rangle = \frac{1}{n(\vec{r},t)} \int v^0 f(\vec{r},\vec{v},t)\, d^3v = 1$$

$$\boxed{\therefore n(\vec{r},t) = \int f(\vec{r},\vec{v},t)\, d^3v} = \frac{\text{no of Particles}}{d^3r} = \frac{\#}{m^3}$$

number density.

(ii) $\chi(\vec{r},\vec{v},t) = v^1$

$$\boxed{\langle v^1 \rangle = \frac{1}{n(\vec{r},t)} \int v^1 f(\vec{r},\vec{v},t)\, d^3v = \vec{u}}$$

macroscopic averge velocity or flow velocity

$$\boxed{m n(\vec{r},t)\,\vec{u}(\vec{r},t) = \int v^1 f(\vec{r},\vec{v},t)\, d^3v = P}$$

momentum density

mean $\chi(\vec{r},\vec{v},t) = v^2$:

varian $\langle v^2 \rangle = \dfrac{1}{n(\vec{r},t)} \int v^2 f(\vec{r},\vec{v},t)\, d^3v = u^2$

~~thermal~~

kinetic energy density
$$\boxed{\frac{1}{2} m\, n(\vec{r},t)\, u^2 = \int v^2 f(\vec{r},\vec{v},t)\, d^3v = R}$$

* Note that: The peculiar velocity or random velocity $\vec{c}$:

the velocity of the particle relative to the average velocity:

$\vec{c} = \vec{v} - \vec{u}$. ; $\langle c \rangle = 0$, $\langle v \rangle = u$

* The pressure:

$$\langle c^2 \rangle = \frac{1}{n(\vec{r},t)} \int c^2 f(\vec{r},\vec{v},t)\, d^3v$$

pressure $\boxed{P = n(r,t) \langle c^2 \rangle}$

* the temperature: is a measure of the mean kinetic energy of the random particle motion.

$$\frac{1}{2} KT = \frac{1}{2} m \langle c^2 \rangle$$

thermal energy

mean kinetic energy of random motion.

* Plasma kinetic equation:

- the plasma kinetic equation:

i- "Compact form": $\frac{Df}{Dt} = C(f) = \left(\frac{\partial f}{\partial t}\right)_{coll}$

ii- Regular form: $\frac{\partial f}{\partial t} + (\vec{v}\cdot\vec{\nabla}_r) f + (\vec{a}\cdot\vec{\nabla}_v) f = C(f)$

iii- Conservative form: $\underbrace{\frac{\partial f}{\partial t} + \vec{\nabla}_r\cdot(\vec{v}f) + \vec{\nabla}_v\cdot(\vec{a}f)}_{\text{long-rang interaction effect}} = \underbrace{C(f)}_{\text{short-range interaction effect}}$

* where:

- $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \vec{a}\cdot\vec{\nabla}$.
- $C(f)$: Collision term
  - → Boltzmann collision operator : Integral.
  - → Fokker-planck " " : differential
  - → Krook " " : Algebra

- $\frac{\partial f}{\partial t}$: accumulation of f with time.
- $(\vec{v}\cdot\vec{\nabla}_r)$: Advection of f with velocity v in configuration space
  : Convection " " " " " " " .
  : Spatial acceleration

- $(\vec{a}\cdot\vec{\nabla}_v)$: Advection of f with velocity a in velocity space.
  : Convection " " " " " " " " .
  : Velocity acceleration.

- $\frac{D}{Dt}\iint d^3r\, d^3v = \int C(f)\, d^3r\, d^3v$

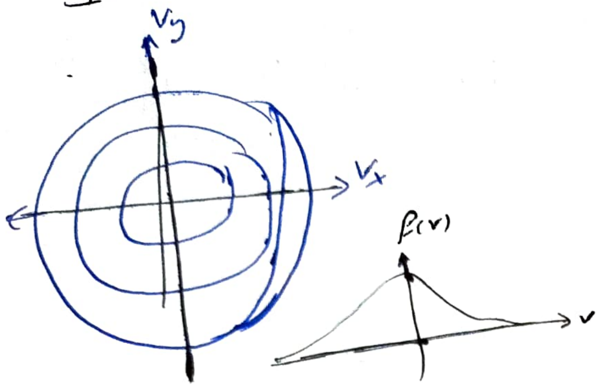
$$\boxed{\frac{DN}{Dt} = N_{coll}}$$

→ Ionization > increases.
→ recombination < decrease.
→ Attachement < decreases.

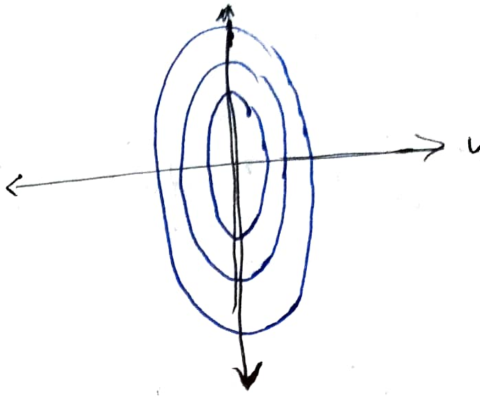
creation anhilation

Types of number distribution function:

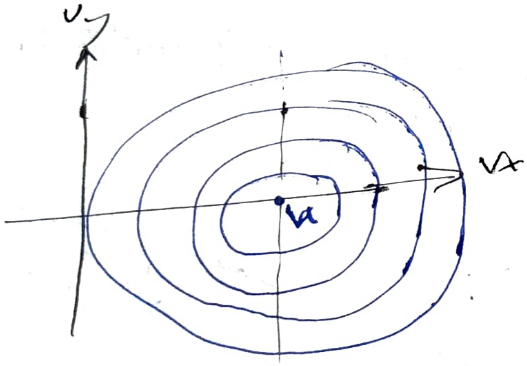
① Isotropic Maxwellian

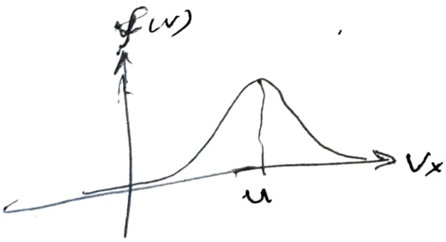


② Anisotropic: $(v_y > v_x)$

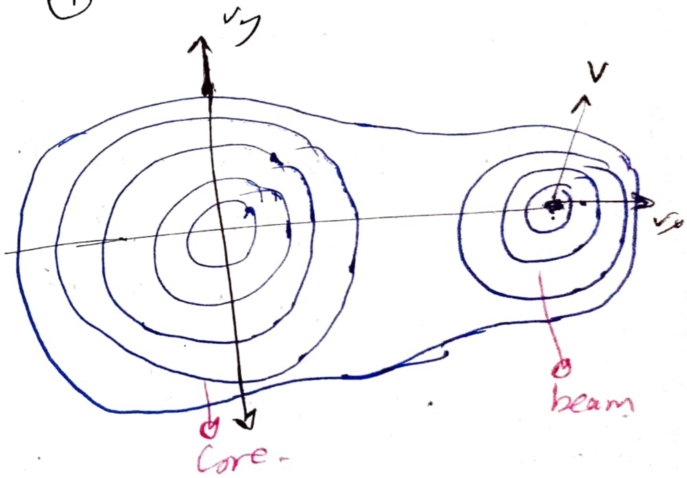


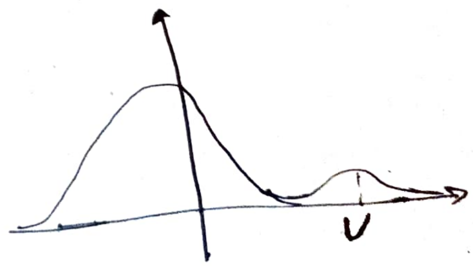
③ Drift-Maxwellian flows





④ Bump on tail / Beam





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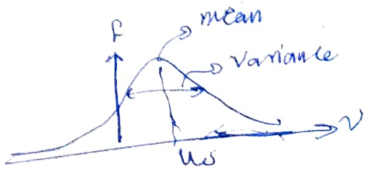
* Moments of distribution function



at $x = x_0$

$$f(x,t) = C e^{-\frac{(u-u_0)^2}{mkT}}$$

Zero moment $\int v^0 dv\, f := C$: Area under

1 moment $\int v\, dv f := n u_0$: Average OR mean value OR

2 moment $\int v^2 dv f = \frac{1}{2} m u_0^2 + P_0 \frac{1}{2} T$ (variance)

3 moment $\int v^3 dv f = Q$: Skewness (symmetry around mean)

* Plasma kinetic eq:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_c = C(f)$$

→ Boltzman collision operator: integral
→ Fokker-Planck collision operator ↳ Different
→ Krook collision operator: algebraic
→ charge exchange collision operator ↳ Integral

$$\underbrace{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dt}}_{\text{long-rang interaction}} = \underbrace{C(f)}_{\text{short-range interaction}}$$

$$\boxed{\frac{\partial f}{\partial t} + (\underline{v}\cdot\underline{\nabla}_x) f + (\underline{a}\cdot\underline{\nabla}_v) f = C(f)}$$

$\frac{\partial f}{\partial t}$: accumulation of f

$(\underline{v}\cdot\underline{\nabla}_x) f$: - advection of f with velocity v in configuration space
- convection in configuration space.
- spatial acceleration.

$(\underline{a}\cdot\underline{\nabla}_v) f$: - advection of f with velocity a in velocity space
- convection term in velocity space.
- velocity acceleration.

- It can be written as:

Creation OR faucet OR formation
annihilation sink OR losses

$$\frac{dN}{dt} = \iint C(f)\, d^3x\, d^3v$$

$$\frac{d}{dt}\iint d^3x\, d^3v\, f = \iint C(f)\, d^3x\, d^3v$$

$$\boxed{\frac{df}{dt} = C(f)}$$