د. إبراهيم القماش

* Fluid Model:

- The ~~full~~ distribution function $f(x, v, t)$ often provide ~~far~~

① far more information than we need.

② far " " than can " " " we can measure.

- So, if we could have governing eqs for the measured quantities. This leads to the fluid moment descriptions of plasma.

* Fluid moments of kinetic plasma equation:

Derivation of fluid eqs

① From microscopic: it is for dilute gas — Boltzmann — not apply for dense gas

② macroscopic derivation: assume the continuum and dence of the gas. not apply for dilute gas.

→ Both gave the same eqs.

$$\frac{df}{dt} = C(f)$$

$$\frac{\partial f}{\partial t} + (\underline{v} \cdot \nabla_r) f + (\overset{F/m}{a} \cdot \nabla_v) f = C(f) \quad ; \quad F = q(E + v \times B) = ma$$

$$\boxed{\frac{\partial f}{\partial t} + (\underline{v} \cdot \nabla_r) f + \left[\frac{q}{m}(E + \underline{v} \times \underline{B}) \cdot \underline{\nabla}\right] f = C(f)}$$ "Regular form"

$$\boxed{\frac{\partial f}{\partial t} + \nabla_r \cdot (vf) + \nabla_v \cdot \overbrace{\left[\frac{q}{m}(E + \underline{v} \times \underline{B}) f\right]}^{(af)} = C(f)}$$ "Conservative form"

* conservation of particles:

$$\frac{\partial f}{\partial t} = -\nabla \cdot (vf) - \nabla_v \cdot \left[\left(\frac{F^{lr}}{m} + \frac{F^{sr}}{m}\right) f\right]$$

$$= -\frac{\partial}{\partial x} \cdot (vf) - \frac{\partial}{\partial v} \cdot \left(\frac{F^{lr}}{m} f\right) - \frac{\partial}{\partial v} \cdot \left(\frac{F^{sr}}{m} f\right)$$

$$= -\frac{\partial}{\partial x} \cdot (vf) - \frac{\partial}{\partial v} (a^{lr} f) - \frac{\partial}{\partial v} \cdot \left(\frac{F^{sr}}{m} f\right)$$

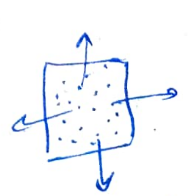
$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \cdot (vf) + \frac{\partial}{\partial v} \cdot (af) = -\frac{\partial}{\partial v} \left(\frac{F^{sr}}{m} f\right) = \left(\frac{\partial f}{\partial t}\right)_c = C(f)$$

$(v \times B) \perp v$

F^{lr}: long range

F^{sr}: short range

OR $$\frac{\partial f}{\partial t} + (v \cdot \nabla_r f) + (a \cdot \nabla_v) f = \left(\frac{\partial f}{\partial t}\right)_c = C(f) \quad (1)$$



* Moments of distribution function $f(\underline{x}, \underline{v}, t)$

density : $n = \int d^3v' f$ → unit of velocity * $\underline{n}$

flow velocity: $\underline{V} = \frac{1}{n} \int d^3v' \underline{v}' f$

flux density: $n\underline{V} = \int d^3v' \underline{v}' f$

u : velocity of ordered motion

C : " " random motion

Pressure: $P = \int d^3v' \frac{1}{2} m v'^2 f$: internal kinetic energy density

heat flux: $q = \int d^3v' \frac{1}{2} m v'^2 \underline{v}' f$

Viscous stress tensor: $\underline{\underline{\Pi}} = \int d^3v' m [\underline{v}'\underline{v}' - \frac{v'^2}{2} \underline{\underline{I}}] f$

particle velocity

$C = V - U$ → average velocity

↳ random velocity

total pressure tensor : $\underline{\underline{P}} = \int d^3v\, m \underline{v}'\underline{v}' = P\underline{\underline{I}} + \underline{\underline{\Pi}}$

$\langle C \rangle = 0$

$\langle v \rangle = u$

total energy flux : $Q = \int d^3v \frac{mv^2}{2} \underline{v} f = q + (\frac{5}{2} P + \frac{1}{2} n m V^2)\underline{V} + \underline{V} \cdot \underline{\underline{\Pi}}$

$\underline{v} = \underline{V} + \underline{v}'$ uu cc

total pressure tensor : $\underline{\underline{P}} = \int d^3v' m \underline{v}\underline{v} f = \int d^3v\, m \underline{V}\underline{V} f + \int d^3v' m \underline{v}'\underline{v}' f$

$= mn\underline{V}\underline{V} + \underline{\underline{P}}$ ← due to random velocity.

$= mn\underline{V}\underline{V} + P\underline{\underline{I}} + \underline{\underline{\Pi}}$

*Collisional contributions:

frictional force : $R = \int d^3v' m \underline{v}' C(f)$

heat frictional force : $F = \int d^3v' m \underline{v} (\frac{mv^2}{2} - \frac{5}{2}) C(f)$

heating : $Q = \int d^3v' \frac{mv'^2}{2} C(f)$

Faucet & sink: $F - S = \int d^3v\, C(f)$

ionization recombination

"Vlasov" eq : describes time-reversible process : $\frac{\partial S}{\partial t} \Rightarrow$ entropy,

*Note that: $\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(vf) + \frac{\partial}{\partial v}(af) = 0$

① Conservation of particles

$\int\int d^3x d^3v$: $\frac{\partial N}{\partial t}$ + 0 + 0 $= 0$ → $\frac{\partial N}{\partial t} = 0$ → N is constant in time

total number.

② conservation of momentum:

$\int\int d^3x d^3v\, m v$: $\frac{\partial P}{\partial t} = 0$; $P = P_{particle} + P_{field} = \text{Const}$

$= \sum m_s v_s + \varepsilon_0 (E \times B)_v$

$\sum \frac{1}{2} m v_s^2 + \frac{\varepsilon_0}{2} E^2 + \frac{B^2}{2\mu_0} = \text{Const}$

③ Conservation of energy

④ Entropy: $S = -\sum \int\int f_s \ln f_s$

$\frac{\partial S}{\partial t} = 0$

$\int\int d^3v d^3x \frac{mv^2}{2}$: $\frac{\partial E}{\partial t} = 0$; $E = E_{particle} + E_{field} = \text{Const}$

... moments: $(\overset{1}{n}, \overset{3}{\underline{u}}, \overset{1}{P})$: 5 Moments

$(\overset{1}{n}, \overset{3}{\underline{u}}, \overset{1}{P}, \overset{3}{q}, \overset{3}{\underline{\Pi}})$: 13 Moment approach

$$\frac{\partial f}{\partial t} + \nabla_r \cdot \underline{v} f + \nabla_v \cdot [q(E + v \times B) f] = C(f) \rightarrow \circledast$$

[1] Matter Conservation (mass density) & Continuity equation (Current balance eq)

$$\int d^3v \, m v^0 \times \left[\frac{\partial f}{\partial t} + \nabla \cdot v f + \nabla \cdot [q(E + v \times B) f] = C(f)\right]$$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \underline{u} = \begin{cases} 0 & ; \int d^3v \, C(f) = 0 \\ F - S & ; \int d^3v \, C(f) = F - S \end{cases}$$

: If $C(f)$: Coulomb collision operator

; else.

[2] Momentum density Conservation & Force density balance eq:

$$\int d^3v \, m \underline{v}^1 \times \left[\frac{\partial f}{\partial t} + \nabla_r \cdot v f + \nabla_v \cdot [q(E + v \times B) f] = C(f)\right]$$

dyadic (2-order tensor)

$$\frac{\partial \rho_m \underline{u}}{\partial t} + \nabla \cdot [\rho_m \underline{u}\,\underline{u} + P \underline{\underline{I}} + \underline{\underline{\Pi}}] = \rho_q (E + \underline{u} \times \underline{B}) + R$$

[3] Energy density conservation & Power density balance

$$\int d^3v \, \frac{m v^2}{2} \times \left[\frac{\partial f}{\partial t} + \nabla_r \cdot v f + \nabla_v \cdot [q(E + v \times B) f] = C(f)\right]$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m u^2 + \frac{P}{\Gamma - 1}\right) + \nabla \cdot \left[\left(\frac{\Gamma}{\Gamma - 1} P + \frac{1}{2}\rho_m u^2\right)\underline{u} + \underline{\underline{\Pi}} \cdot \underline{u} + q\right] = \rho_q E \cdot \underline{u} + \underline{R} \cdot \underline{u} + Q$$

$\frac{1}{\Gamma - 1} = \frac{3}{2}$ $\qquad \frac{\Gamma}{\Gamma - 1} = \frac{5}{2} = \frac{3}{2} + 1$

$$\frac{\partial P}{\partial t} + u \cdot \nabla P = -\Gamma P \nabla \cdot v + (\Gamma - 1)[-\nabla \cdot q + \Pi : \nabla v + \eta J^2]$$

q : heat flux (?)

① Continuity eq:

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \underline{u} = \begin{cases} 0 \, ; \\ F - S \end{cases}$$

* Notes: it state " how the matter transfort from one location to another".

⇒ $\rho_m = mn$: mass density $\equiv \frac{\text{mass}}{\text{Volume}}$; $n = \frac{N}{V}$ number density

⇒ $\frac{\partial \rho_m}{\partial t}$ = mass Current $\equiv \frac{\partial q}{\partial t} = I$: electric Curret

⇒ $\underline{u}$: flow velocity of the fluid element.

⇒ $\rho_m u \equiv J_m$ = flux density $\equiv$ mass Current density. $\equiv J_q = \rho_q \underline{u} = nq\underline{u}$ electric Curent density.

Differential form

$$\Rightarrow \quad \frac{\partial \rho_m}{\partial t} + \nabla \cdot J_m = \begin{cases} 0 \\ F - S . \end{cases}$$

$$\frac{\partial M}{\partial t} + \oint_S J_m \cdot dA = 0$$

$$\frac{\partial}{\partial t} \int_V \rho_m \, dv + \int_V \nabla \cdot J_m \, dv = 0$$

States that: " transport of the matter from one location to another."

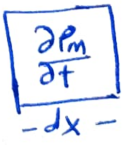
(I) ⓘ if $= 0 \Rightarrow$ - the matter is Conserved in the system.
- the no of particles " " "

ⓘⓘ if $= F - S \Rightarrow$ - the matter is not Conserved due to (Faucet - Sink).
- " no of Particles " " "

ionization (creation) & Recombination & (annihilation)

[2] $\left. \frac{\partial \rho_m}{\partial t} \right|_x \mp \left(\nabla \cdot (\rho_m u) \right) \Big|_x = 0$

- the change of amount of matter in a volume d^3x at specific location x ~~is due to~~ $\left(\frac{\partial \rho_m}{\partial t} \right)$

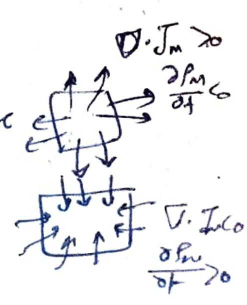


is due to

- the net flux of the matter from this volume.

① if $\nabla \cdot J_m > 0$ (Divergence) ⇒ $\frac{\partial \rho_m}{\partial t} < 0$ ⇒ amount of matt decrease

② " $\nabla \cdot J_m < 0$ (Convergence) ⇒ $\frac{\partial \rho_m}{\partial t} > 0$ ⇒ " " " increase



Momentum density Conservation/ Force density balance:

$$\underbrace{\frac{\partial}{\partial t}(\rho_m \underline{u})}_{\text{momentum density}} + \nabla\cdot\Big(\overbrace{\rho_m \underline{u}\,\underline{u} + P\underline{\underline{I}} + \underline{\underline{\Pi}}}^{\text{momentum density flux}}\Big) = \rho_q\left(\underline{E} + \underline{V}\times\underline{B}\right) + \underline{R}$$

Notes

- momentum density: $\rho_m u = n m \underline{u} = P_m$
- inertia force density: $\frac{\partial \rho_m u}{\partial t}$; ρ_m : nF.
- $\rho_m \underline{u}\,\underline{u}$: external momentum flux.
  - " in longitudinal direction.
- $P\underline{I}$: internal "
- $\underline{\underline{\Pi}}$: Viscosity tensor OR shear-stress tensor.
- ρ_q = charge density = nq : density → $q>0$: acceleration
- $\rho_q(\underline{E}+\underline{V}\times\underline{B})$: Lorentz force → $q<0$: deceleration.
- R : collisional friction force density {elastic, inelastic}
- $\nabla\cdot P\underline{I} = \nabla P$: pressure thermal/internal gradiant force density.
- $\nabla\cdot\underline{\underline{\Pi}}$: Viscosity force density
- $\nabla\cdot(\rho_m \underline{u}\,\underline{u})$: Kinetic force density.
- It Can be written as:

$$\frac{\partial \rho_M}{\partial t} + \nabla\cdot J_M = F_M - S_M \;;\quad \begin{aligned} \rho_M &= \rho_m \underline{u} \\ J_M &= \rho_m \underline{u}\,\underline{u} + P\underline{I} + \underline{\underline{\Pi}} \\ F_M &= \rho_q(E + V\times B) \\ S_M &= R \end{aligned}$$

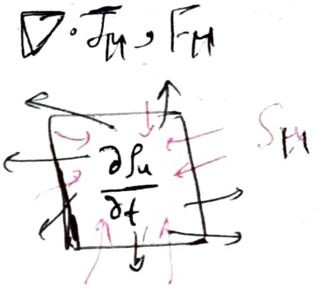
- It states:

- It governs how the momentum density transport from one location to another

- It is a force density balance eq.:

- $\frac{\partial \rho_M}{\partial t} = -\nabla \cdot J_M + F_M - S_M$

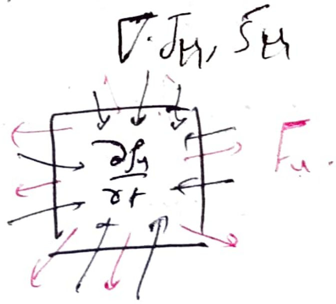
(i) If $\nabla \cdot J_M > 0$: acceleration



$\frac{\partial \rho_u}{\partial t} > 0$ if $\nabla \cdot J_M > 0$, $F > 0$,

$F_u > S_u + \nabla \cdot J_u$

(ii) If $\nabla \cdot J_M < 0$:



$\frac{\partial \rho_u}{\partial t} < 0$: if $\nabla \cdot J_M < 0$, $F > 0$

$\nabla \cdot J_M + F_M < S_u$

- Deceleration

Energy density conservation / Power density balance:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m u^2 + \frac{P}{\Gamma-1}\right) + \nabla\cdot\left[\left(\frac{\Gamma}{\Gamma-1}P + \frac{1}{2}\rho_m u^2\right)\underline{u} + \underline{\underline{\Pi}}\cdot\underline{u} + \underline{q}\right]$$

$$= \rho_q \underline{E}\cdot\underline{u} + \underline{R}\cdot\underline{u} + Q$$

Notes

- external energy density: $\frac{1}{2}\rho_m u^2$
- Internal " " : $\frac{P}{\Gamma-1}$
- Total energy density: $\rho_E = \frac{1}{2}\rho_m u^2 + \frac{P}{\Gamma-1}$
- Power density: $\frac{\partial \rho_E}{\partial t}$.
- Internal Energy flux: $\frac{\Gamma}{\Gamma-1} P \underline{u}$
- External " " : $\frac{1}{2}\rho_m u^2 \underline{u}$
- shear " " " : $\underline{\underline{\Pi}}\cdot\underline{u}$
- Heat energy ~~density~~ flux: $\underline{q}$.
- internal Power density: $\nabla\cdot\frac{\Gamma}{\Gamma-1} P \underline{u}$ ← Convection flow
- External Power " : $\nabla\cdot\frac{1}{2}\rho_m u^2 \underline{u}$
- viscous Power density: $\nabla\cdot(\underline{\underline{\Pi}}\cdot\underline{u})$
- heat Power density: $\nabla\cdot\underline{q}$ ← Conduction flow.
- Electrical Power density: $\rho_q \underline{E}\cdot\underline{u}$
- Collisional " " : $\underline{R}\cdot\underline{u}$
- Heat Conduction Power density: Q ← heat friction Power.

- It can be written as:

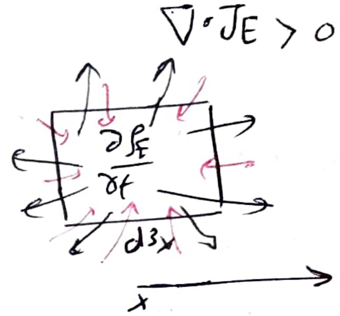
$$\frac{\partial \rho_E}{\partial t} + \nabla \cdot J_E = F_E - S_E .$$

- It states governs how the energy density transport from one location to another.

- It is a power density balance eq.

power at this location

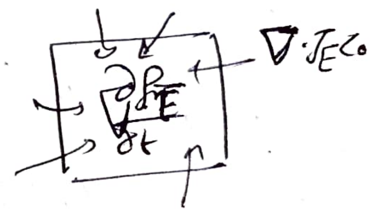
① $\frac{\partial \rho_E}{\partial t} \uparrow$ increases if



$\nabla \cdot J_E < 0$

$F_E > 0$

$S_E \cancel{>} 0$



$F_E > S_E + \nabla \cdot J_E$

② If $\frac{\partial \rho_E}{\partial t} \cancel{<} \phi$: power decreases if

$\nabla \cdot J_E \cancel{<} 0$

$F_E > 0$

$S_E > 0$

$F_E \cancel{-} \nabla \cdot J_E < S_E .$

...se eqs can be written as: "Conservative form"

"Eulerian picture"

$\Rightarrow$ A volume element is fixed in space in the "laboratory" frame of reference.

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot J_i = F_i - S_i$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho_m \\ \rho_u \\ \rho_E \end{pmatrix} + \nabla \cdot \begin{pmatrix} J_m \\ J_u \\ J_E \end{pmatrix} = \begin{pmatrix} F_m - S_m \\ F_u - S_u \\ F_E - S_E \end{pmatrix}$$

$\leftarrow$ mass density $\rho_m = nm$

$\leftarrow$ momentum density $\rho_u = \rho_m u$

$\leftarrow$ Energy density $\rho_E = \rho_m \frac{u^2}{2}$

$$\frac{\partial}{\partial t}\begin{pmatrix} \text{quantity} \\ \text{density} \end{pmatrix} + \nabla \cdot \begin{pmatrix} \text{quantity} \\ \text{flux} \\ \text{OR} \\ \text{quantity} \\ \text{current} \\ \text{density} \end{pmatrix} = \begin{pmatrix} \text{Quantity} \\ \text{faucet} \\ - \\ \text{Quantity} \\ \text{Sink} \end{pmatrix}$$

- Conservation of
  - $\rightarrow$ matter (mass, energy, number)
  - $\rightarrow$ momentum
  - $\rightarrow$ Energy
- Continuity of
  - $\rightarrow$ current density $\frac{\partial \rho_m}{\partial t}$
  - $\rightarrow$ Force " $\frac{\partial \rho_m u}{\partial t}$
  - $\rightarrow$ Power " $\frac{\partial E\rho_m}{\partial t}$
- transport of
  - $\rightarrow$ ~~momentum~~ mass density ρ_m
  - $\rightarrow$ momentum " $\rho_m u$
  - $\rightarrow$ Energy density $\rho_m E$
- Balance of
  - $\rightarrow$ current density
  - $\rightarrow$ Force density
  - $\rightarrow$ Power density

$J_m = \rho_m u$: mass flux.

$J_u = \rho_m \underline{u}\,u$: momentum flux

$J_E = E\,\underline{u}$: energy flux

- It governs: v^0 v^1 v^2 : Velocity moments

(1) Conservation of : matter / ~~tra~~momentum / energy ---

(2) transport of : matter / momentum / energy --

(3) Balance of : Current density / Force density / Power density ---

(4) Continuity of : mass density / momentum density / Energy density

* General transport equation:

$$\frac{\partial}{\partial t}(n\langle\chi\rangle) + \nabla\cdot(n\langle\chi\vec{v}\rangle) - n\langle\vec{a}\cdot\nabla_v\chi\rangle$$
$$= \left[\frac{\delta}{\delta t}(n\langle\chi\rangle)\right]_{coll}$$

(i) Continuity, $\chi = m$

(ii) momentum $\chi = m\vec{v}$, $\vec{V} = \vec{c} + \vec{u}$

(iii) Energy $\chi = \frac{mV^2}{2}$

* Fluid equations in "Regular" form: "Lagrangian picture"

"the Surface of the volume element is Co-moving with the fluid, in the fluid frame of reference".

$$\frac{dn}{dt} = -n(\nabla \cdot V)$$

$$mn\frac{du}{dt} = nq(\underline{E}+\underline{V}\times\underline{B}) - \nabla P - \underline{\nabla}\cdot\underline{\Pi} + R$$

$$\frac{1}{\Gamma-1} n \frac{dT}{dt} + P\nabla\cdot\underline{u} = -\nabla\cdot q - \underline{\Pi}:\nabla\underline{v} + Q$$

Convection flux; Conduction flux; Viscous dissipation; Collisional heating

$$Tn\frac{ds}{dt} = -\nabla\cdot q - \underline{\nabla u : \Pi} + Q$$

entropy production rate; heat conduction; viscous dissipation; collisional heating; Sources of entropy production

* closures:

Pont; $R \sim mn\nu\underline{V} \sim nq\frac{\underline{J}}{\sigma}$; $q \sim -k\nabla n$; $\Pi \sim \eta\nabla\underline{v}$

* the lagrangian derivative: (Material derivative (convection derivative).

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V\cdot\nabla$$

$V\cdot\nabla$ → advective derivative. OR convective derivative

Note: It is called advection in horizontal direct & convection " vertical direct.

* Euler (lab frame & field picture)

fluid flow as seen by an observer in the lab frame at fixed position $\left(\frac{\partial F(x,t)}{\partial t}\right)$,

* Lagrangian (Parcel frame, Particle picture).

fluid flow as seen by an observer sitting on a fluid parcel. flow with the fluid element. $\left(\frac{dF(x,t)}{dt}\right)$

Lagrangian ← $\frac{d}{dt} = \frac{\partial}{\partial t} + V\cdot\nabla$ ↙ Euler

Euler: $\frac{\partial}{\partial t}$

Lagrange: $\frac{d}{dt}$

- Physical Side : theory : (Postulates)
- Practical " : model (Assumptions)

CM theory

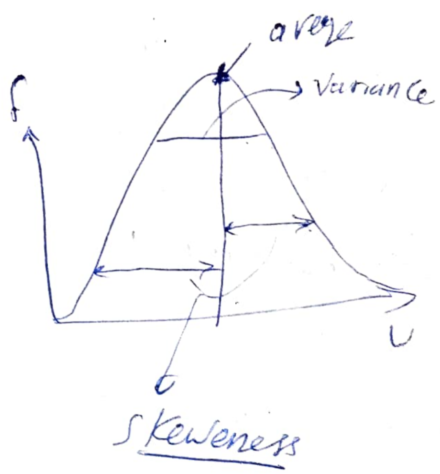
Newtonian model → Force fn → F

Lagrangian model → Lagrangian function → L

Hamiltonian model → Hamiltonian function → H

(averges):
* Moments of distribution function

v^0 : Area : total number.
v^1 : averge. : momentum
v^2 : Variance. : Energy
v^3 : skeweness. : Heat conduction.



Plasma theory:

SPM → Individual approach "CM"

KM → Statistical approach "SM"

FM → averges approach "FM"