إبراهيم الكماش

Concept of Instability

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- Instability: a general way of redistributing free energy which has accumulated in a ~~non eq~~ a non-equilibrium state.

Equilibrium → static $F=0$ → Dynamic $\sum_j F_j = 0$

Stability analysis → Particle analysis → Field wave analysis?

- Types of mechanical ~~equilibrium~~ equilibrium ← particle

Stability analysis for mechanical system (particle)

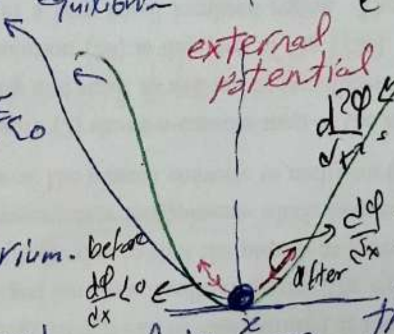
① stable equilibrium: bottom of the trough potential function Equilibrium State

graphically ← $F = -\frac{dV}{dx}$ → algebrically

(i) $\frac{dV}{dx} = 0 \Rightarrow F = 0$ equilibrium

(ii) $\frac{d^2V}{dx^2} > 0 \Rightarrow \frac{dF}{dx} < 0$

stable equilibrium.



external potential

$\frac{+ - (-)}{+ - (-)} \neq 0$ $\frac{dV}{dx} = 0$ (equilibrium)

$\frac{d^2\varphi}{dx^2} = \frac{\frac{d\varphi}{dx}\big|_{after} - \frac{d\varphi}{dx}\big|_{before}}{x_{after} - x_{before}}$ $\frac{dV}{dx} \neq 0$ (non-equilibrium)

before $\frac{d\varphi}{dx} < 0$; after $\frac{d\varphi}{dx} > 0$

Stability: $\frac{dF}{dx} = -\frac{d^2V}{dx^2}$ (i) < 0 : stable point : Force decrease ; (ii) > 0 : unstable point : Force increase with x.

- The sphere lie on the lowest point of an infinitly high potential trough. In this position the sphere can only perform oscillations around its equilibrium position, which will damp out due to friction until the sphere comes to rest at the bottom of the potential trough.

(iii) $\frac{d^2V}{dx^2} = 0$ → inflection point

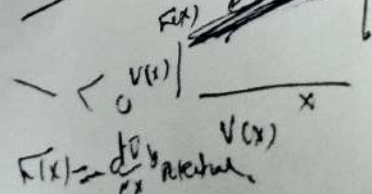
$\frac{d^2\varphi}{dx^2} = \frac{\frac{d\varphi}{dx}\big|_{after} - \frac{d\varphi}{dx}\big|_{before}}{x_{after} - x_{before}}$

top of the hill

bottom of the valley.

$\frac{(+) - (-)}{+ - (+)} > 0$

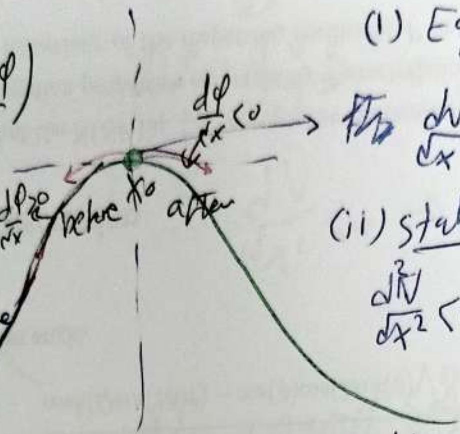
$\frac{d^3V}{dx^3} > 0$; < 0



$F(x) = -\frac{d\varphi}{dx}$ potential

① =

linear instability: top of a potential hill.



$$\left.\frac{d^2\varphi}{dx^2}\right|_{x=x_0} = \frac{d}{dx}\left(\frac{d\varphi}{dx}\right) = \frac{\left(\frac{d\varphi}{dx}\right)_{after} - \left(\frac{d\varphi}{dx}\right)_{before}}{x_{after} - x_{before}} = \frac{(-)-(+)}{(+)-(+)} < 0$$

(i) Equilibrium

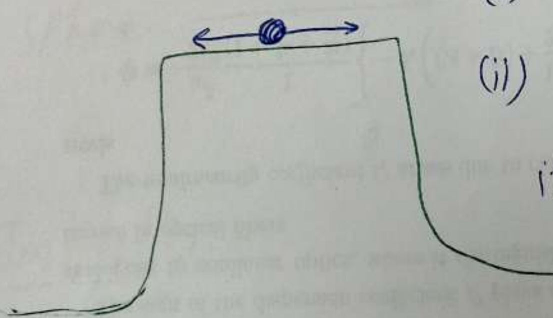
$\frac{dV}{dx} = 0 \Rightarrow F = 0$ equilibrium

(ii) stability:

$\frac{d^2V}{dx^2} < 0 \Rightarrow \frac{dF}{dx} > 0 \Rightarrow$ unstable.

- The slightest linear perturbation of its position, will let it roll down the hill and will not come back to the equilibrium position again.

(iii) Meta stable state: a plateau on top of a hill:



(i) $\frac{dV}{dx} = 0 \Rightarrow F = 0$

(ii) $\frac{d^2V}{dx^2} = 0 \Rightarrow \frac{dF}{dx} = 0$

it has many equilibrium positions.

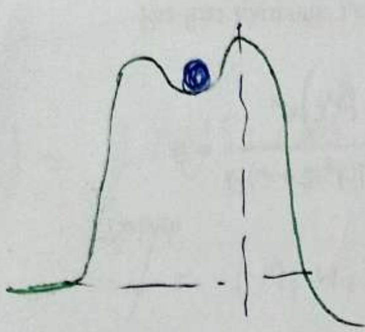
- the sphere can wander around until it reaches the crest and rolls down.

nonlinear instability:

(i) $\frac{dV}{dx} = 0 \rightarrow F = 0$ equilibrium.

(ii) $\frac{d^2V}{dx^2} \geq 0 \Rightarrow \frac{dF}{dx} \leq 0$ at $x < x_c$ stable

$\frac{d^2V}{dx^2} < 0 \Rightarrow \frac{dF}{dx} > 0$ at $x > x_c$ unstable.



- The sphere is stable against small-amplitude perturbati but becomes unstable for large amplitude

* Analogy with plasma:

- potential $\longrightarrow$ Free energy
- sphere $\longrightarrow$ Wave or mode. system

* Instability:

(i) Large scale macroinstability:

changes the spatial gradiant and inhomogenities

(ii) Small-scale microinstabity:

changes the local distribution function.

How we can study linear instability in plasma?

[stability analysis for waves (Fields)]

"Linear analysis"

1- physical quantities.

$$f = (n, u, \phi, A, P, E, B \dots)$$

2- Equilibrium state (fixed points):

$$f_0 = (n_0, u_0, \phi_0, A_0, P_0, E_0, B_0, \dots)$$

3- Consider the perturbation for the ~~waveform~~

$$f = f_0 + \varepsilon \tilde{f}_* \quad ; \quad \tilde{f} \ll f_0 \text{ (linear)}.$$

4- Consider the perturbation in the form of plane wave function

$$\tilde{f} = \tilde{f}_0 \, e^{i(kx - \omega t)}$$

5- We don't know ω & k of the wave.

We need a determinant equation of ω & k.

So, we need a determinant equation of ω & k.

which is the dispersion relation

$$\omega = \omega(k; \text{plasma parameters})$$

wave — k; medium — plasma parameters.

Classifications of ω and the perturbed amplitude $\tilde{f}$:

$$\tilde{f} = \tilde{f}_0\, e^{i(kx-\omega t)}$$

$\tilde{f} \rightarrow$ perturbed amplitude

Let:

$$\omega = \omega_R + I\,\omega_I$$

$$\boxed{\tilde{f} = \tilde{f}_0\, e^{\omega_I t}\, e^{i(kx-\omega_R t)}}$$

$$\tilde{f} = \tilde{f}_0\, e^{i(kx-\omega_R t)}$$

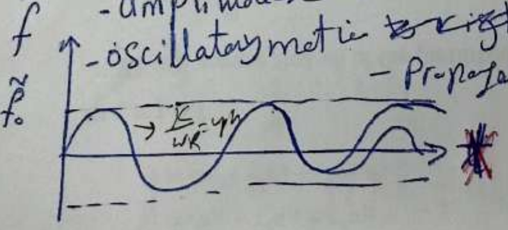
$\tilde{f}$ = time dependent amplitude * phasor

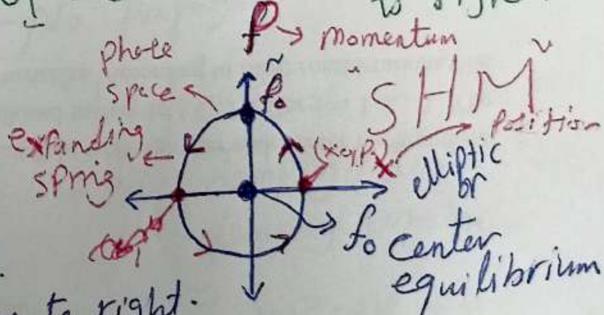
(i) $\omega = \omega_R > 0$ & $\omega_I = 0 \longrightarrow$ oscillatory motion to right.

$$\tilde{f} = \tilde{f}_0\, e^{i(kx-\omega_R t)}$$

momentum

- amplitude $\rightarrow$ Const.
- oscillatory motion ~~to right~~.
- Propagating to right.



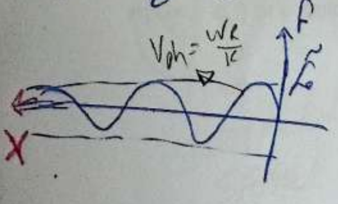


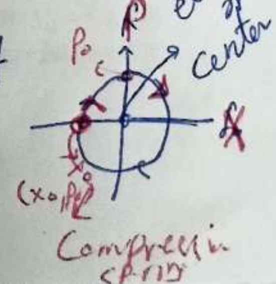
(ii) $\omega = \omega_R < 0$ & $\omega_I = 0 \longrightarrow$ oscillatory motion to left

$$\tilde{f} = \tilde{f}_0\, e^{i(kx+\omega_R t)}$$

- Constant amplitude
- oscillatory motion ($\omega_R \neq 0$) ~~to left~~
- propagating to left ($k \neq 0$)

"SHM"





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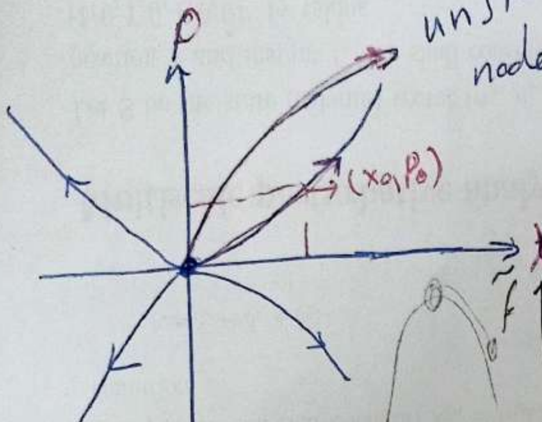
- $\omega = \omega_I > 0$, $\omega_R = 0 \rightarrow$ growing non-oscillatory motion.
standing wave

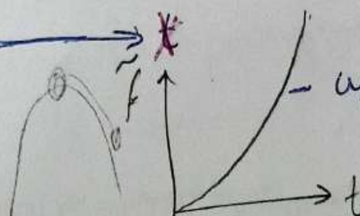
$\tilde{f} = \tilde{f}_0 e^{i\omega_I t} e^{iKx}$

- amplitude (momentum) increases exponentially with time.
- No oscillation in time ($\omega_R = 0$)
- oscillation in space ($K \neq 0$)
- No propagation ($\omega_R = 0$)

("F H O")

unstable node





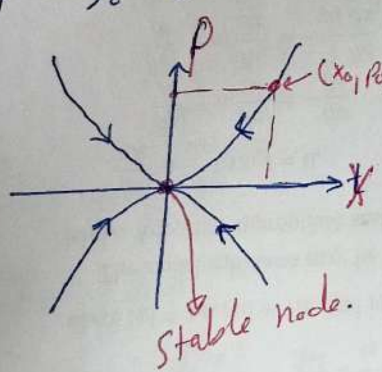
- ω_I : growth rate.
: growth time.

iv) $\omega = \omega_I < 0$; $\omega_R = 0 \longrightarrow$ damping non-oscillate

$\tilde{f} = \tilde{f}_0 e^{-\omega_I t} e^{\mp iKx}$ standing wave

- amplitude (momentum) decreases exponentially with time.
- No oscillation in t ($\omega_R = 0$)
- oscillation in space ($K \neq 0$)
- No propagation ($K \neq 0$ & $\omega_R = 0$)

("D H O")

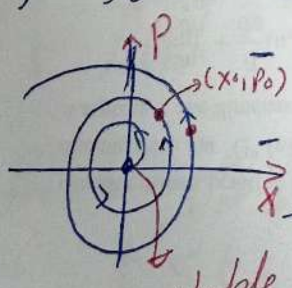


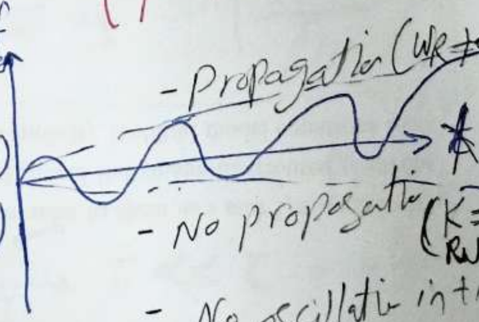
- ω_I : damping rate.
: damping time.

(v) $\omega = \pm\omega_R \pm i\omega_I$ & $\omega_I > 0$ K & $\omega_R \neq 0$ propagation

$\tilde{f} = \tilde{f}_0 e^{\omega_I t} e^{i(Kx \mp \omega_R t)}$ ("F H M")

- amplitude (momentum) increases with time.
- oscillatory ($\omega_R \neq 0$)
- propagating ($K \neq 0$)
to right (−)
left (+)





- Propagation ($\omega_R \neq 0$ & $K \neq 0$)
- No propagation ($K = 0$, $\omega_R = 0$)
- No oscillation in time ($\omega_R = 0$)
- No " " space ($K = 0$)

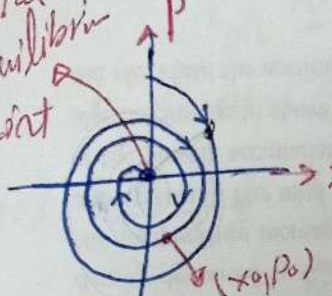
$\omega = \omega_R + I\omega_I$; $\omega_I < 0$

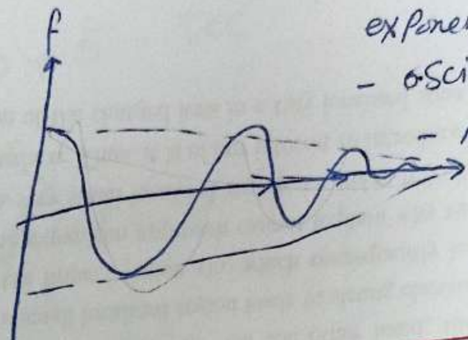
$$\tilde{f} = \tilde{f}_0 e^{-\omega_I t} e^{i(kx \mp \omega_R t)}$$ ("DHM")

- amplitude (momentum) decrease exponentially with time.
- oscillatory ($\omega_R \neq 0$)
- propagating ($K \neq 0$)
  (-) left
  (+) Right.

Stable spiral equilibrium point

Phase space





* Breakdown of linear analysis:

"Instability saturates at a certin amplitude due to the effect of the nonlinear terms, instead of growing indefinitly as suggested by linear theory".

- $$\tilde{f} = \underbrace{\tilde{f}_0 e^{\omega_I t}} e^{i(kx - \omega_R t)}$$

$$\tilde{f} = \hat{f}(t) e^{i(kx - \omega_R t)}$$

$$\hat{f}(t) = \tilde{f}_0 e^{\omega_I t}$$

- The Condition

$$\frac{\hat{f}(t)}{\tilde{f}_0} \ll 1$$

if $t = t_{NL} \rightarrow \hat{f}(t) = \tilde{f}_0$ equilibrium

$$\ln\left(\frac{\hat{f}_0(t)}{\hat{f}(t)}\right) = \omega_I t_{NL} \longrightarrow \boxed{t_{NL} \approx \omega_I^{-1} \ln\left(\frac{\tilde{f}_0}{\hat{f}(t)}\right)}$$ perturbed

for linear approximation

time scale of wave

(i) $t < t_{NL}$ & $\omega \gg \gamma\omega_I \rightarrow t \lll \tau \rightarrow$ linear $\rightarrow \tilde{f} \ll \tilde{f}_0$

(ii) $t > t_{NL}$ & $\omega_I \gg \omega \rightarrow \tau \gg t \longrightarrow$ Nonlinear analysis $\hat{f} \sim \tilde{f}_0$

mple:

(i) $\cancel{f(t)}\, f_0 = 1$, $\hat{f}(t) = 0.9$

$\omega_I = 2\,s^{-1}$

$\therefore t_{NL} \simeq 0.05$ sec.

$t_{w} \leq 0.05$ sec

(ii) $f_0 = 1$; $\hat{f}(t) = 0.9$

$\omega_I = 10$

$t_{NL} \simeq 0.01$ sec.

$t_{w} \leq 0.01$

(iii) $f_0 = 1$; $\hat{f}(t) = 0.1$

$\omega_I = 2$

$t_{NL} \simeq 1.15$ sec.

$t_{w} \leq 1.15$ sec

(iv) $f_0 = 1$; $\hat{f}(t) = 0.1$

$\omega_I = 10$

$t_{NL} \simeq 0.2$ sec.

$t_{w} \leq 0.2$ sec.

wave

So,

Energy stored in a electrostatic wave :

$$W_\omega(\omega,k) = \frac{\epsilon_0}{2}\langle|E(\omega,k)|^2\rangle \frac{\partial}{\partial\omega}\left[\omega\,\epsilon(\omega,k)\right]$$

$$= W_E \frac{\partial}{\partial\omega}\left[\omega\,\epsilon(\omega,k)\right]$$

wave energy density.

$$\frac{W_\omega}{W_E} = \frac{\partial}{\partial\omega}\left[\omega\,\epsilon(\omega,k)\right] \simeq \omega\frac{\partial\epsilon}{\partial\omega}.$$

W_E ↙ electostatic energy density

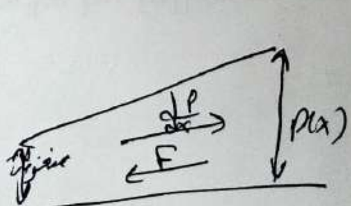
stability analysis

Particle → oscillation analysis

Field → wave analysis → $f \sim f_0 e^{i(kx-\omega t)}$

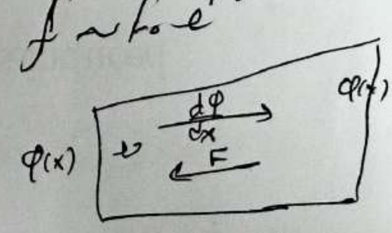
$F = -\frac{d\phi}{dx}$ ← Potential

$F = -\frac{dP}{dx}$ ← pressure.



$\frac{dP}{dx} \longrightarrow >0$

$F \longleftarrow <0$



$\varphi(x)$: scaler

$\frac{d\varphi}{dx} \longrightarrow >0$

$F \longleftarrow <0$

Stability analysis:

Particle

analysis

CM

mechanical

Basic they

Field

Wave analysis

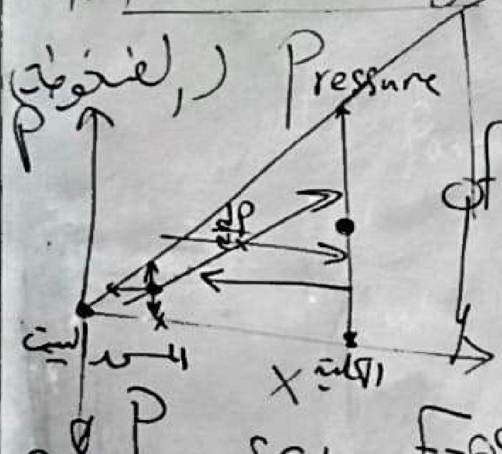
Plasma

GR

Advanced they

* Particle analysis

$\vec{\nabla} P \rightarrow$ scalar



$f = \ominus k x$

P scalar

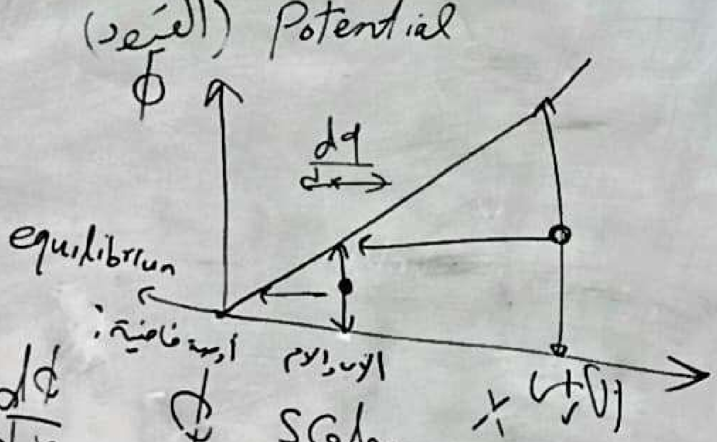
$F = \ominus \frac{dP}{dx}$

$\frac{dP}{dx} > 0 \longrightarrow$

$F \longleftarrow$

$f = -\frac{d\phi}{dx}$

restoring force



ϕ scalar

$\frac{d\phi}{dx} > 0 : \longrightarrow$

$F < 0 : \longleftarrow$

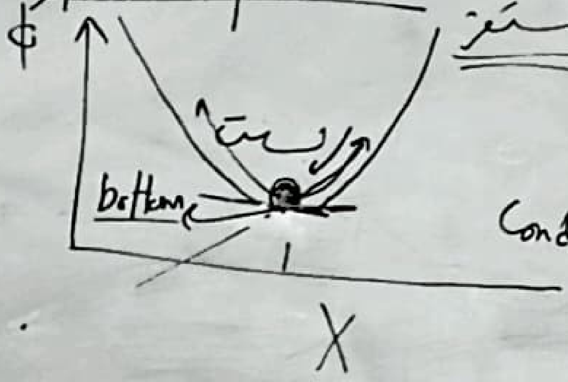
* Particle anal..

equilibrium

Stable → اتزان مستقر

unstable → اتزان غير مستقر

(i) Stable equilibrium:



(i) equilibrium:

Condition:

$$F = -\frac{d\phi}{dx} = 0$$

(ii) $\frac{d\phi}{dx} > 0$: Stable equilibrium.

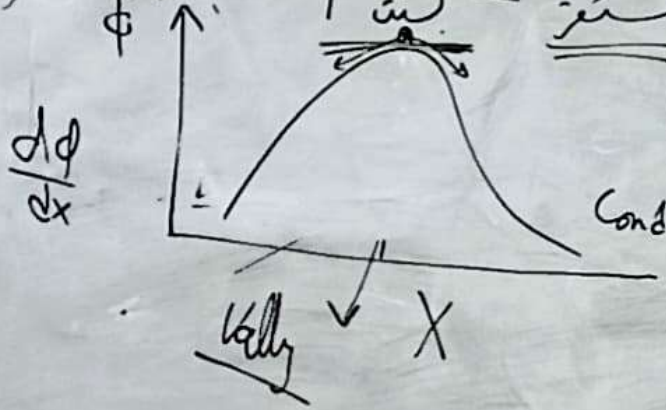
* Particle an...

equilibrium

stable — ازان مستقر

unstable — ازان غير مستقر

(ii) : un stable equilibrium:



(i) equilibrium:

Condition: $F = -\frac{d\phi}{dx} = 0$

(i) $\frac{d\phi}{dx} = 0$: condition equilibrium.

(ii) $\frac{d^2\phi}{dx^2} < 0$: un-stable equilibrium.

* Field analysis ← wave

Constant → stable

damping → Stable equilibrium

* Amplitude

growing → unstable equilibrium

السعة

الثابت

[illegible]

Field

Space