ch(8) Dispersion

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(i)
- If the wave speed depends only on the physical properities of the medium (i.e. elastic and inertia...), then the wave speed is constant, independent of its frequency or wavelength. Dispersion: the dependence of the wave velocity on (i) width of the pulse OR (ii) wavelength of the wave.
- Such medium is called a non-dispersive medium and wave travelling through this medium will maintain a constant shape (wave form).
- Since, the wave speed is constant, all frequencies travel at the same speed and the pulse maintain a constant shape. OR width.

-Dispersion → velocity depend on wavelength.
-Nonlinearity → " " " amplitude.

(ii) If the wave speed depends on the frequency of the wave. this medium is called dispersive medium. The wave with higher frequency will travel diffent than the lower frequency. This leads to distortion of the waveform (width OR ~~amplitude~~ frequency) Broadening chirping high and low frequencies

separation waves of different wavelengths into short and long wavelength waves.

change wavelength not frequency

- So, dispersion: the dependence of wave speed on frequency OR λ
frequency amplitude OR " " refractive index on " OR λ.
- Dispersion relation: the relation between frequency ω and wavenumber $k \Rightarrow \omega = \omega(k;$ medium properities

properities of wave energy momentum

it shows the medium properitities

* Examples: rainbow, separation of white light into colors in prism

Types of medium:

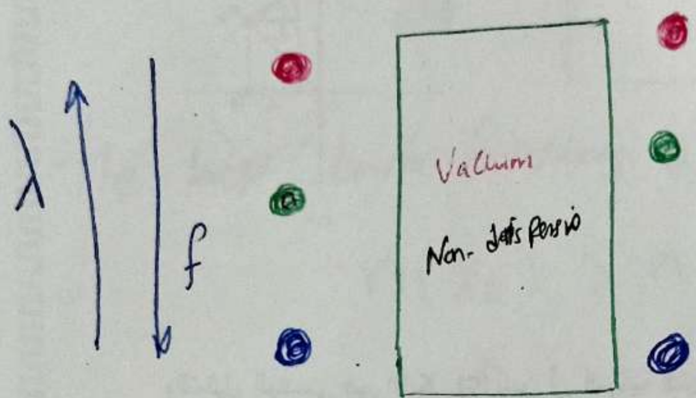
Assume three photons; Red light Photons + Green light Photon + Blue light photon

enters different medium, who is the fastest?

① Non-dispersive medium (vacuum):

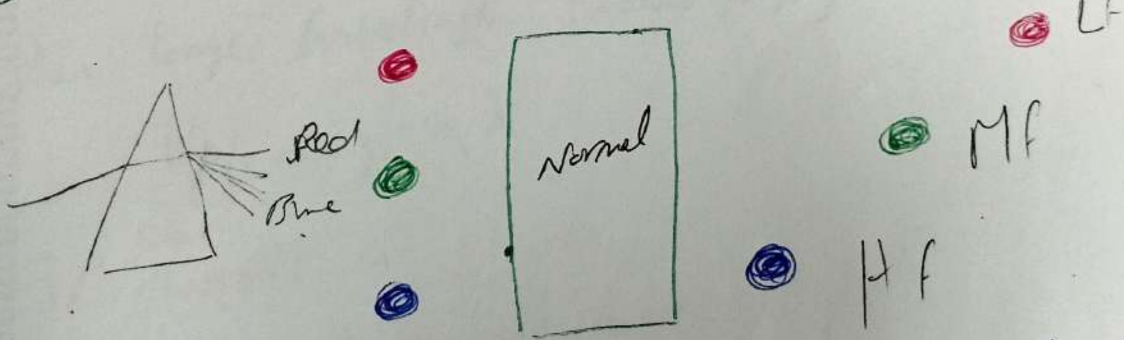
$n(\lambda_{red}) = n(\lambda_{Green}) = n(\lambda_B)$

v_{ph} doesn't change and doesn't depends on frequency.



Exp. EMW in vacuum.

② Normal-dispersive medium (air):



Cauchy's law $n(\lambda_{red}) < n(\lambda_G) < n(\lambda_B) \Rightarrow n \propto \frac{1}{\lambda}$

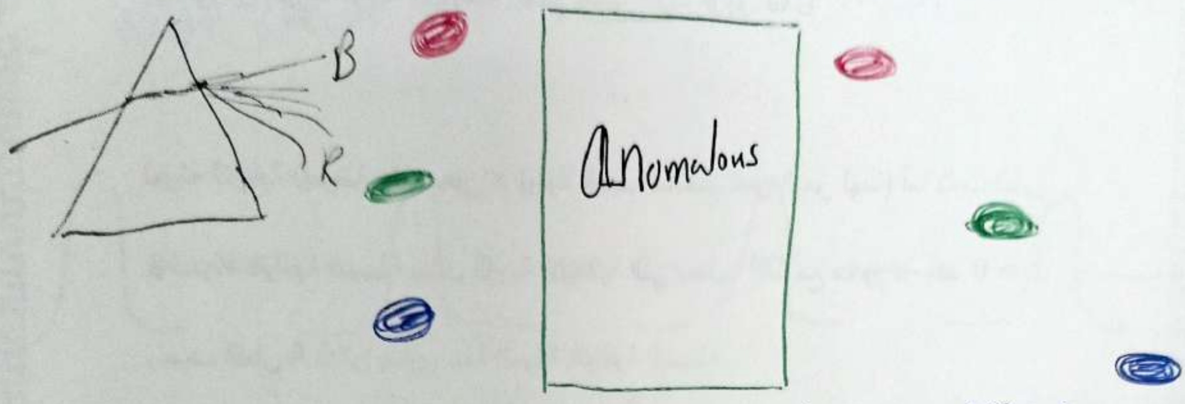
- Longer wavelength (lower frequency) moves faster than shorter wavelength (higher frequency) $\Rightarrow v_{ph} \propto \cancel{\lambda}$

$v_{ph} \propto \frac{1}{k}$

Exp: Sound wave in air.

Gravity wave in deep water.

Anomalous dispersive medium (Plasma):



- The ~~larger~~ lower frequency will disperse more.

$$n(\lambda_R) > n(\lambda_G) > n(\lambda_B)$$

$$n \propto \lambda .$$

- Shorter wavelength (higher frequency) moves faster than longer wavelength (lower frequency).

$$V_{ph} \propto \frac{1}{\lambda} \propto k .$$

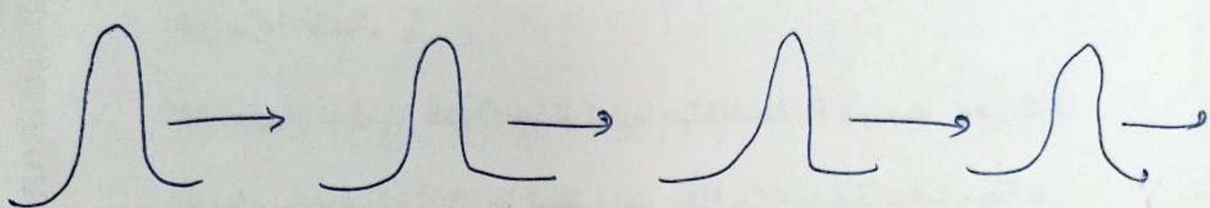
EXP: Some waves in plasma
Capillary wave in shallow water.

	Non	Normal	Anomalous
Condition	$V_{ph} = const$	$V_{ph} \propto \lambda$ $V_{ph} > V_g$	$V_{ph} \propto \frac{1}{\lambda}$ $V_g > V_{ph}$
Consequences	Long wavelength speed = Short wavelength speed	longer > shorter longer faster	Longer < shorter Longer slower

Types of waves:

① Individual wave: monochromatic wave (Single frequen)

i- Wave speed is constant: waveform presene.

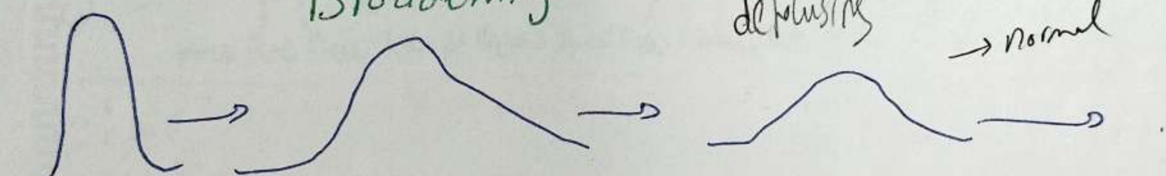


ii- Wave speed depends on frequency: Distortion

Broadening

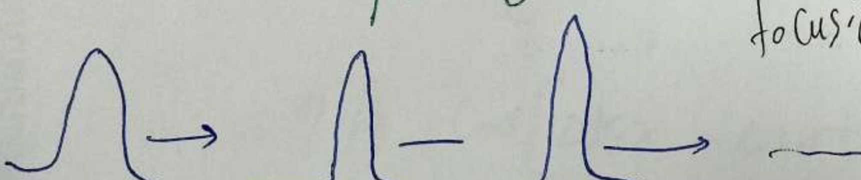
defocusing

→ normal



Pulsing.

focusing → anomalous



Wavepacket (group): (Interference & Modulation)

- Consider ~~two~~ the superposition of two individual waves:

$$y_1 = A\cos(k_1 x - \omega_1 t)$$

$$y_2 = A\cos(k_2 x - \omega_2 t)$$

$$y = 2A\cos\left[\left(\frac{k_2-k_1}{2}\right)x - \left(\frac{\omega_2-\omega_1}{2}\right)t\right]\cos\left[\left(\frac{k_1+k_2}{2}\right)x - \left(\frac{\omega_2+\omega_1}{2}\right)t\right]$$

$$\boxed{y = A(x,t)\,\cos[k_0 x - \omega_0 t]}$$

$A(x,t)$ → envelope; $\cos[k_0 x - \omega_0 t]$ → Carrier wave

1- Amplitude modulation (AM)
2- Frequency " (FM)
3- Phase " (PM)

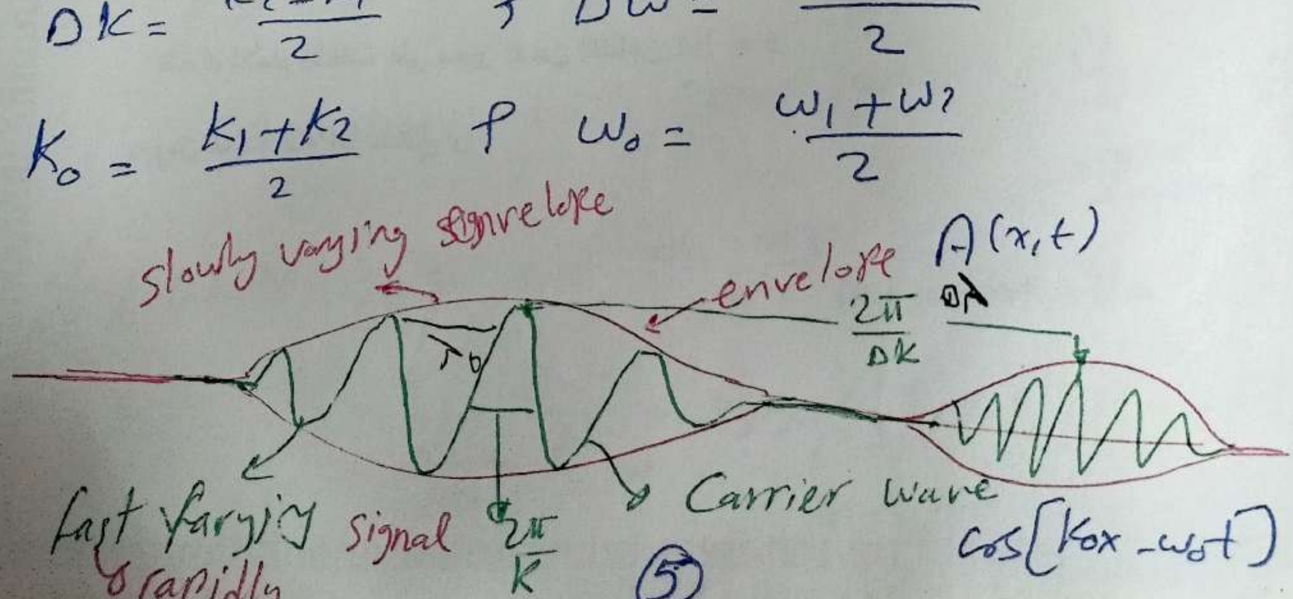
where

$$A(x,t) = 2A\cos\left[\left(\frac{k_2-k_1}{2}\right)x - \left(\frac{\omega_2-\omega_1}{2}\right)t\right]$$

$$= 2A\cos[\Delta k x - \Delta\omega t]$$

$$\Delta k = \frac{k_2-k_1}{2} \quad \& \quad \Delta\omega = \frac{\omega_2-\omega_1}{2}$$

$$k_0 = \frac{k_1+k_2}{2} \quad \& \quad \omega_0 = \frac{\omega_1+\omega_2}{2}$$



The individual waves with same phase velocity

$$V_{ph_1} = \frac{\omega_1}{k_1} \quad \& \quad V_{ph_2} = \frac{\omega_2}{k_2}$$

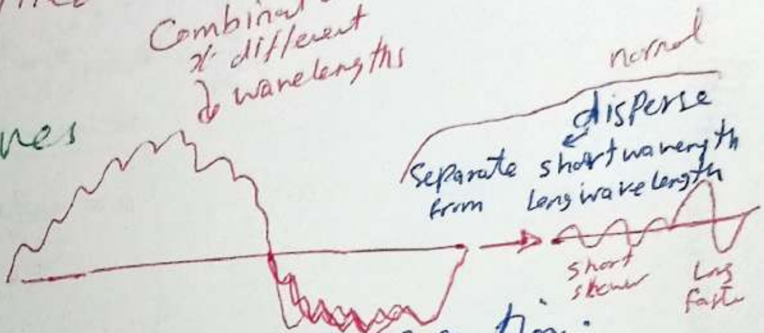
$$V_{ph_1} = V_{ph_2}$$

- wavepacket doesn't change its shape.

$\Rightarrow$ No-dispersive. medium

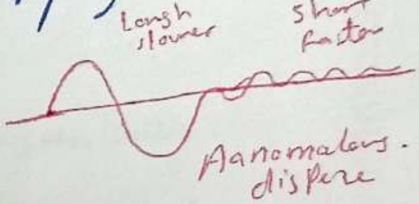
(ii) The individual waves

$$V_{ph_1} \neq V_{ph_2}$$



- wave packet distors during propagation.

- Dispersive medium -



* Phase velocity vs group velocity:

- Carrier wave moves with phase velocity V_{ph}:

$$\frac{d}{dt}(k_0 x - \omega_0 t) = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega_0}{k_0} = V_{ph}.$$

- Envelope moves with group velocity V_g:

$$\frac{d}{dt}(\Delta k\, x - \Delta\omega t) = 0 \Rightarrow \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} = V_g.$$

$\leftarrow$ discrete frequencies

$$\lim_{\substack{\Delta\omega \to 0 \\ \Delta k \to 0}} \frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k} = V_g$$

Continuum frequencies

1) Dispersion relation and types of dispersion:

- $V_{ph} = \frac{\omega}{k}$ & $V_g = \frac{d\omega}{dk}$

$$V_g = \frac{d}{dk}(k V_{ph}) = V_{ph} + k\frac{dv}{dk} = V_{ph} + k\frac{dv_{ph}}{d\lambda}\frac{d\lambda}{dk}$$

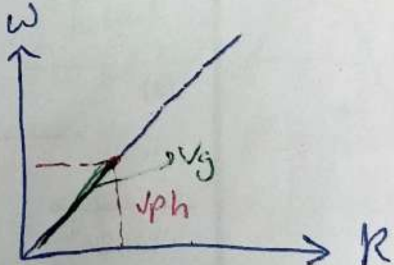
since: $k = \frac{2\pi}{\lambda}$ & $\frac{d\lambda}{dk} = -\frac{\lambda}{k}$

$$\boxed{\begin{aligned} \therefore\ V_g &= V_{ph} + k\frac{dv}{dk} \\ V_g &= V_{ph} - \lambda\frac{dv_{ph}}{d\lambda} \end{aligned}}$$

(i) Non dispersive:

$\frac{dv_{ph}}{d\lambda} = 0 \Rightarrow V_g = V_{ph}$

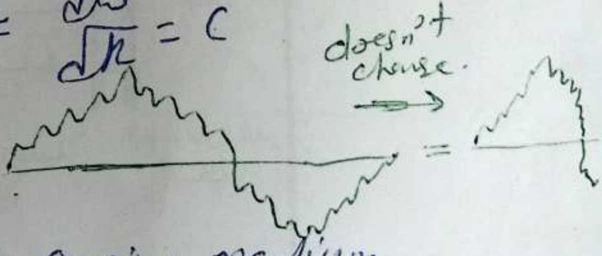
$\frac{dv_{ph}}{dk} = 0$



$\omega = ck$ EMW in vacuum.

$V_{ph} = \frac{\omega}{k} = c$ & $V_g = \frac{d\omega}{dk} = c$

$\therefore V_g = V_{ph}$



∴ the vacuum is non-dispersive medium

- EMW ~~is~~ ~~non~~ doesn't change.
- Sound wave $\omega = \sqrt{\frac{\gamma P}{\rho}}\, k$
- String $\omega = \sqrt{\frac{T}{\mu}}\, k$

Normal-dispersive: $GVD = \frac{d^2\omega}{dk^2} > 0$ (down-chirping)

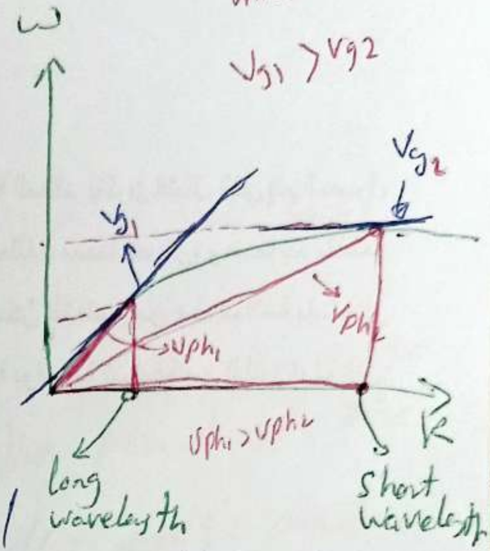
$v_{ph1} > v_{g1}$

$v_{ph2} > v_{g2}$

$v_{g1} > v_{g2}$

$\frac{dv_{ph}}{d\lambda} > 0$ & $v_{ph} > v_g$

$\frac{dv_{ph}}{dk} < 0 \Rightarrow v_{ph} > v_g$



Sound wave

÷ IAW in plasma

- Wave is stiffness spring.

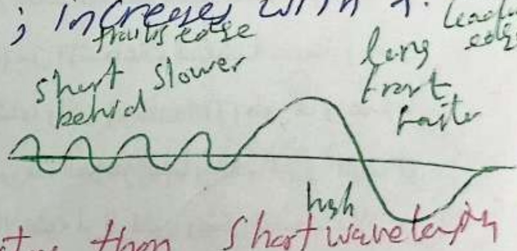
$\omega^2 = \frac{k^2}{1+k^2} \Rightarrow \frac{\omega}{k} = \frac{1}{\sqrt{1+k^2}} \Rightarrow v_{ph}$ decrease with k.

- $\omega^2 = \omega_p^2 + c^2k^2$: EMW in plasm

$v_{ph} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}} > c$ & $v_g = \frac{c^2}{v_{ph}}$

(i) v_{ph} decreases with k ; increases with λ.

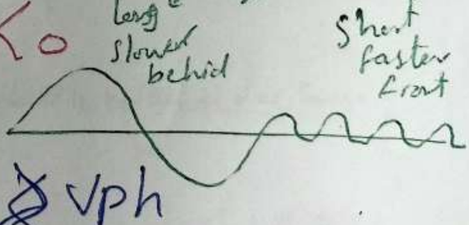
(2) Always $v_{ph} > v_g$.



- long wavelength (lower freq) wave is faster than short wavelength (high)

- the individual wave with v_{ph} in the trailing edge catch the wave group in the leading edge.

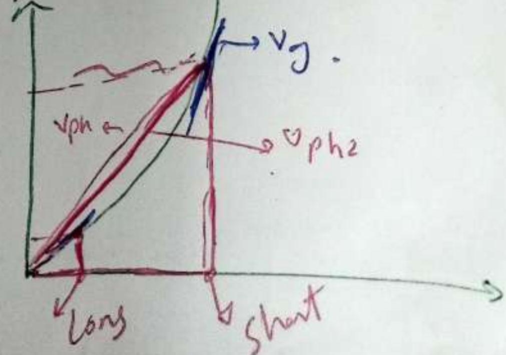
(ii) Anomalous: $GVD = \frac{d^2\omega}{dk^2} < 0$ (up-chirping)



$\frac{dv_{ph}}{d\lambda} < 0 \Rightarrow \frac{dv_{ph}}{dk} > 0 \rightarrow v_g > v_{ph}$

$\omega^2 = ak^2 + bk^3$

$\frac{\omega}{k} = \sqrt{a+bk}$

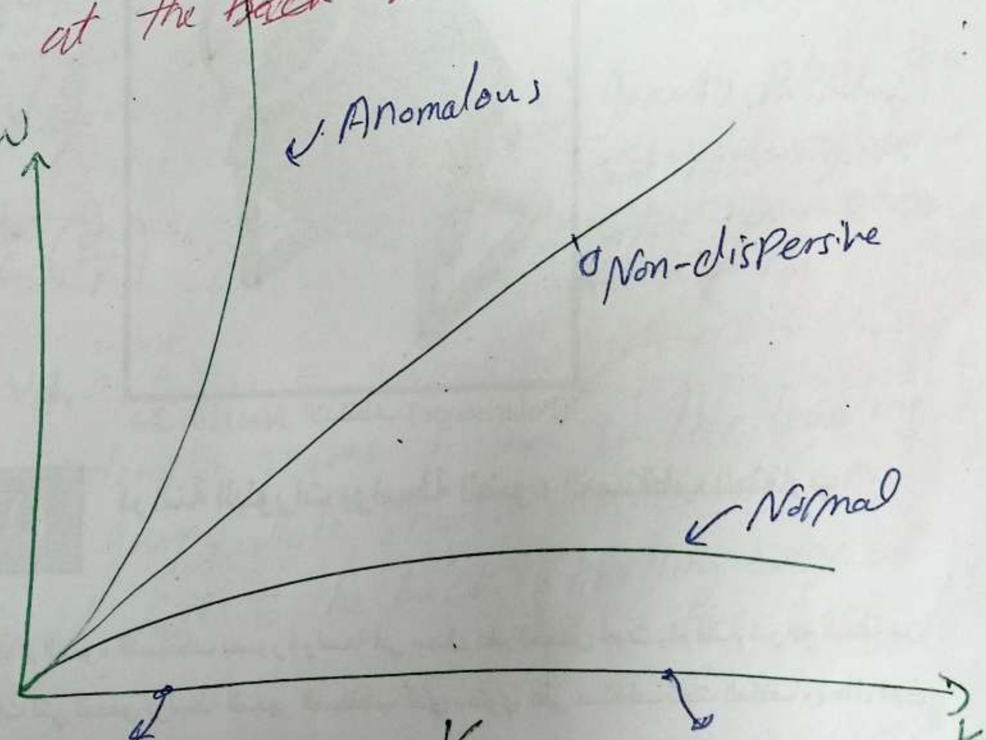


(1) v_{ph} increases with k.

(2) $v_g > v_{ph}$.

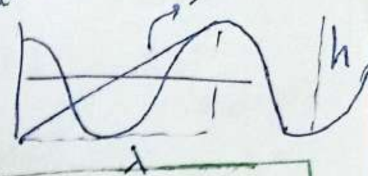
i.e. - Shorter wavelength is faster than ~~short~~ longer.

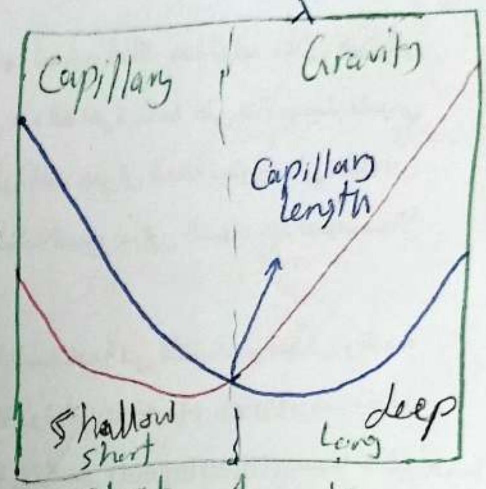
- The individual wave v_{ph} at the front of the leading edge group, moves toward the leading edge and disappear at the ~~back~~ trailing edge of the group.



Gravity-Capillary Waves:

$$\omega^2 = \left(\overset{\text{Gravity}}{gk} + \overset{\text{Surface tension}}{\frac{\sigma}{\underset{\text{density}}{\rho}}} k^3\right) \overset{\text{Wave steepness}}{\tanh(hk)}$$





(1) Long ~~Short~~ wavelength regime or deep water

Gravity ~~Capillary~~ wave: $kh \gg 1$

velocity V_g

$\tanh(hk) \sim \cancel{hk}\ 1$ V_{ph}

$$K^2 \leq \frac{\rho g}{\sigma}$$

$$h \gg \lambda$$

$$\omega = \sqrt{gk}$$

$$V_{ph} = \sqrt{\frac{g}{k}} \sim k^{-\frac{1}{2}} \propto \sqrt{\lambda}$$

$$\sqrt{\lambda} \sim V_g = \frac{1}{2}\left(\frac{g}{k}\right)^{\frac{1}{2}} \sim \frac{1}{2} V_{ph}$$

④ gravity plays the dominant restoring force and surface tension plays a minor role ⑤

$$V_{ph} = 2 v_g \leftarrow \text{Normal-dispersive.}$$

① Longer wavelength moves faster to the leading edge (front) of the group, while shorter wavelength are slower and fall to the back (trailing edge of the group).

② The phase speed of individual wave appears at the back of the group, increases and ~~finally disappears~~ catch up with the front of the group and finally disappear $V_{ph} > V_g$.

③ the phase velocity ~~of~~ for gravity wave is twice the group velocity.

Short wavelength or Capillary wave: $k^2 \gg \frac{\rho g}{\sigma}$

Shallow water $kh \ll 1$

$\tanh(kh) \sim kh$. h depth.

"$h \ll \lambda$"

depth wavelength

$$\omega^2 = \sqrt{\frac{\sigma h}{\rho}}\, k^2 \leftarrow \text{Kelvin equation.}$$

$$V_{ph} = \frac{\omega}{k} = \sqrt{\frac{\sigma h}{\rho}}\, k \propto \frac{1}{\lambda}$$

$$V_g = \frac{3}{2}\sqrt{\frac{\sigma h}{\rho}}\, k \sim \frac{3}{2} V_{ph} \propto \frac{1}{\lambda}$$

anomalous dispersion.

(1) Thus the group velocity of Capillary wave is 1.5 times the phase velocity.

(2) In this regime, gravity plays minor role while the surface tension is the dominant restoring force

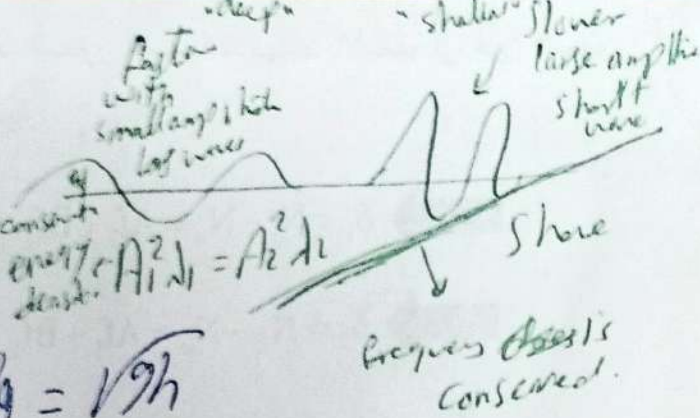
(3) The shorter wavelength waves move faster to the front of the group, while the long wave length waves are slower and fall back to the group.

(4) The group speed faster than the ~~speed~~ phase speed. i.e. the individual wave at the front of the group moves towards to the back, due to the group moves forward, disappearing at the back.

$\sigma = 0$:

* shallow water

$$\omega = \sqrt{gh}\, k$$



$$v_{ph} = \sqrt{gh} \quad \& \quad v_g = \sqrt{gh}$$

$\therefore v_g = v_{ph}$: Non-dispersive → all wavelengths travel at same speed.

* Deep water is the same.

- At gravitation energy = surface energy

$$\frac{U_s}{U_g} = \frac{k^2 \sigma}{g\rho} = 1$$

$$\therefore k = \left(\frac{g\rho}{\sigma}\right)^{\frac{1}{2}} \Rightarrow \xi = \left(\frac{\sigma}{g\rho}\right)^{\frac{1}{2}}$$ capillary length

- for pure water $g = 9.8\ m/sec^2$, $\rho = 10^3\ kg$; $\sigma = 0.072\ \frac{N}{m}$.

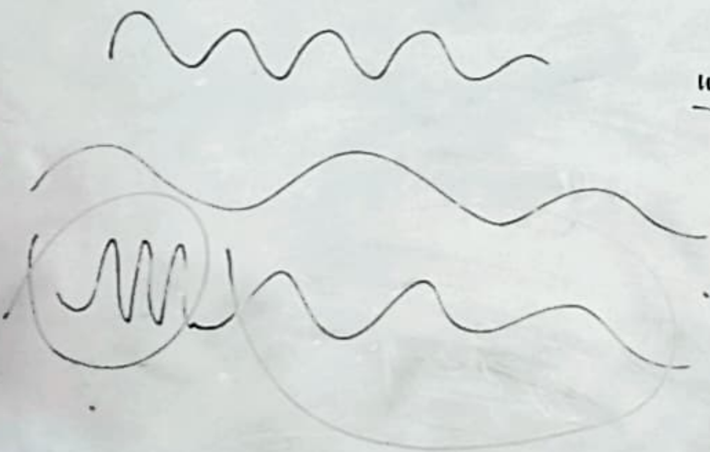
$k \simeq 3.7\ cm^{-1}$ & $\lambda = 1.7\ cm$

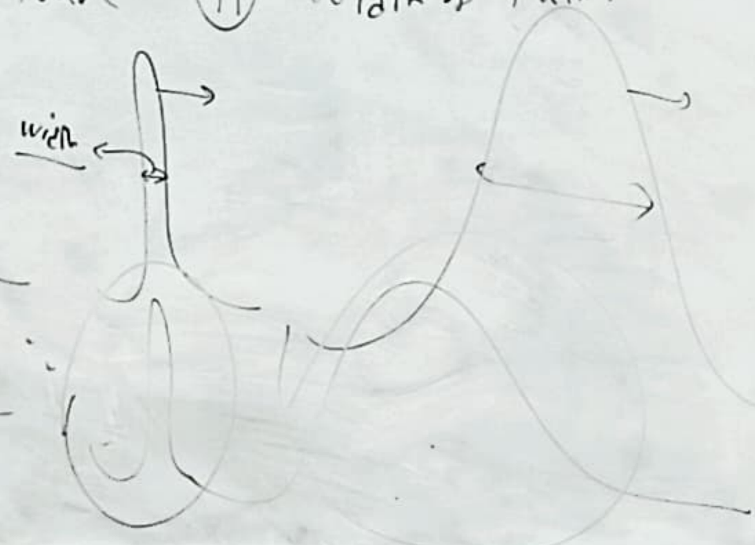
$\xi = 0.27\ cm \Rightarrow$ Gravity = Surface role.

Dispersion

* Dispersion: the dependence of wave velocity:

(i) Wavelength of wave (ii) Width of Pulse.

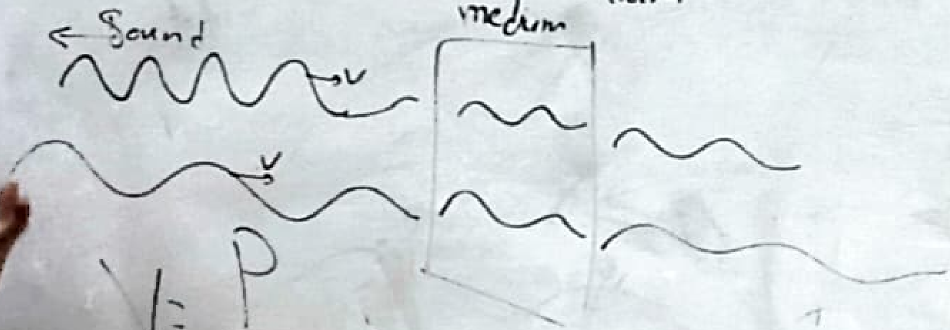




Dispersion

* Types dispersion:

(1) Non-dispersive — wave / medium   exp: Soundwave in air

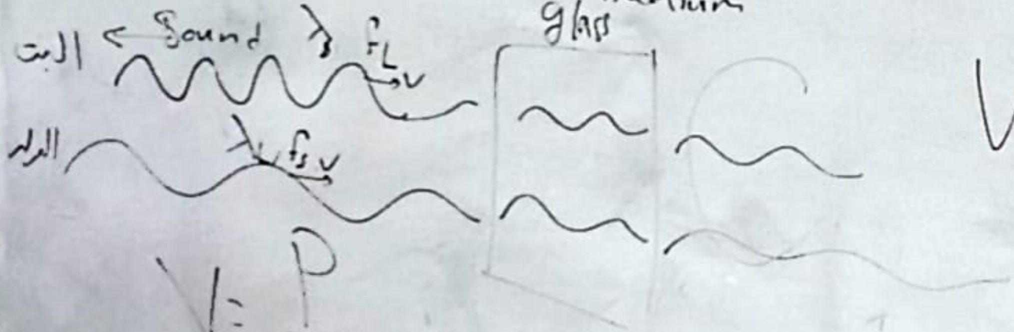


$$V = \frac{P}{\rho}$$

Dispersion

* Types dispersion:

(1) Normal dispersive < wave / medium exp: Sound wave in air



$$V = \frac{P}{\rho}$$

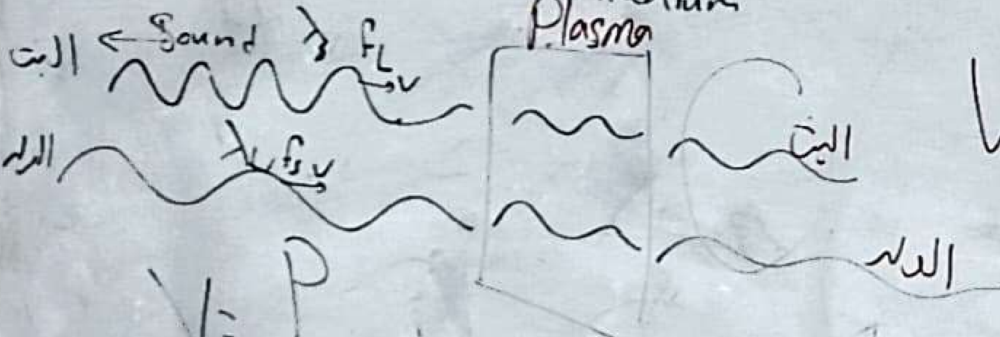
$$V \propto \lambda \propto \frac{1}{f}$$

Dispersion

* Types dispersion:

(1) الشاذ Anomalous dispersive — Wave / medium

exp: Soundwave in air



$V = \frac{P}{\rho}$

$V \propto \frac{1}{\lambda}$ البنت > الولد

$\propto f$

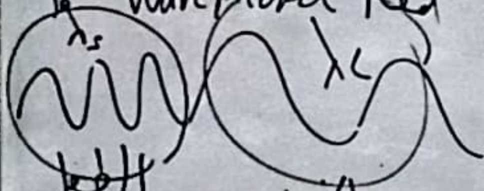
Dispersion

$V \propto \lambda$

* Types dispersion

wave packet Red

λ_s λ_L



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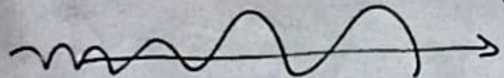


Medium

Anomalous

λ_s الازب

λ الولد

$$\frac{V}{\lambda} = \text{Const}$$

$V_s \propto \lambda_s$

$V_L \propto \lambda_L$

Dispersion

	Non	Normal	Anomalous
Condition	$V \neq$ Const $= \lambda f$	$V \alpha \lambda$ $\alpha \frac{1}{f}$	$V \alpha \frac{1}{\lambda}$ αf
Consequences	Velocity of Longer wavele low f $=$ Short wavelength higher f	Velocity of Longer / low $>$ Shorter, higher	Velocity of Longer / low $<$ Shorter, higher

Dispersion

* Dispersion Relation

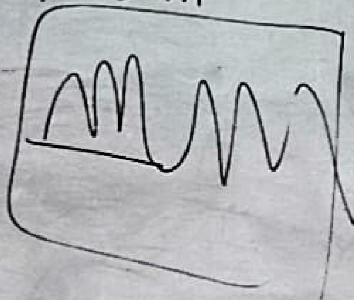
طول $\leftarrow \lambda$, $f \xrightarrow{t}$

Laplace transf $\Leftarrow$ k , ω fourier transf

عدد الاهتزازات
في
الثانية

عدد الموجات
في
الطول

$K = 3 \frac{}{m}$



Dispersion

* Dispersion Relation

Examples

① Light in glass

DR $\boxed{\omega = cK}$ $\Rightarrow \dfrac{\omega}{K} = v_{ph} = c = \text{Const}$

* Dispersion Relation

Dispersion

Normal

$v \propto \lambda$

$\propto \frac{1}{k}$

Example

① Light in glass

DR $\boxed{\omega = cK + \frac{a}{k}}$ $\Rightarrow v_{ph} = \frac{\omega}{k} = c + \frac{a}{k^2}$

v_{ph}

Dispersion

* Dispersion Relation

Examples

① Light in Plasma

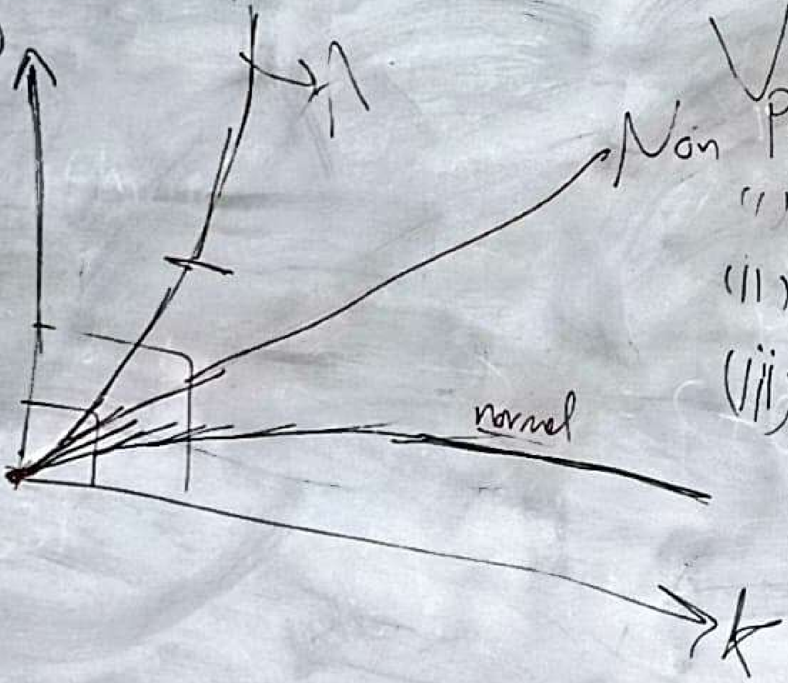
DR

$$\boxed{\omega^2 = c^2k^2 + \alpha k^3} \Rightarrow \frac{\omega^2}{k^2} = c + \alpha k$$

$$v_{ph} = \sqrt{c + \alpha k}$$

Dispersion

* DR



$V_{ph} = \frac{\omega}{k} = \text{slope}$

(i) $= \text{Const}$

(ii) slope $V_{ph} \propto \frac{1}{k}$ Normal

(iii) $V_{ph} \propto k$: anomalous