Physics Between The Beauty of Experimentation and The Magnificence of Ideas

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A farmer, it is said, heard a team of scientists to advice him on improving his dairy production. After six months work, they prepared their report. The farmer begin to read, only to encounter the opening sentence

"Consider a spherical COW …."

What is this Lecture about?...

- What we mean by Paradigm of science ?

A paradigm shift is a fundamental change in the basic concepts and experimental practices of a scientific discipline

It is a concept in the philosophy of science that was introduced and brought into the common by Thomas Kuhn

The scientific method

I Law: Law of Inertia ----- COUNTER-INTUITIVE

What is 'equilibrium'?

Relative to whom?
- frame of reference

Equilibrium means 'state of rest', or of uniform motion along a straight line.

Equilibrium sustains itself, needs no cause; determined entirely by initial conditions.

$$\vec{F} = m\vec{a}$$

Effect $\vec{a}$ is proportional to the Cause $\vec{F}$.

Proportionality: Mass/Inertia.

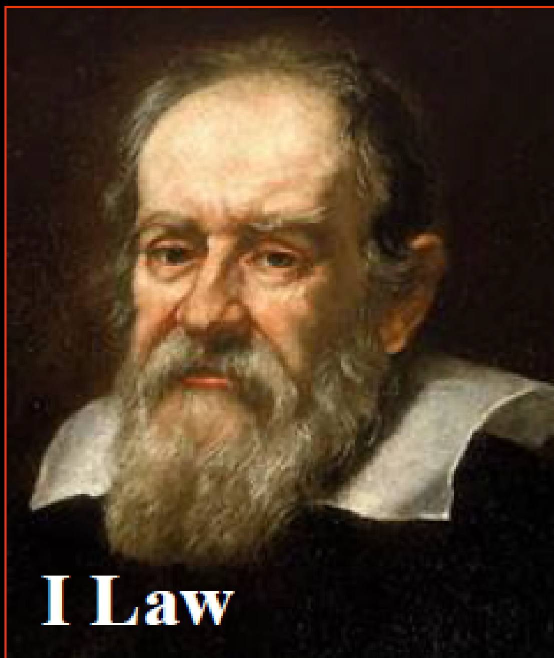
Linear Response.

Principle of causality/determinism.

Galileo; Newton

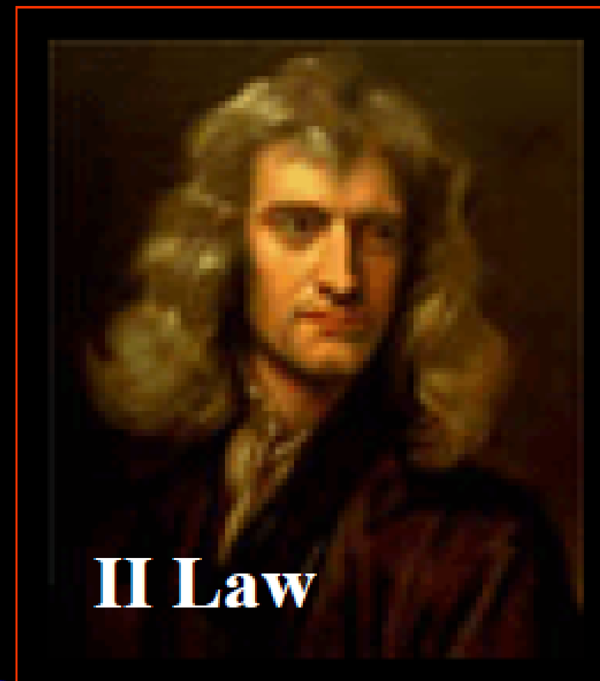
What is 'equilibrium'?
What causes departure from 'equilibrium'?

Galileo Galilei
1564 - 1642



Causality
&
Determinism

Isaac Newton
(1642-1727)



$\vec{F} = m\vec{a}$ *Effect* is proportional to the *Cause*.

Linear Response. Principle of causality.

Force: Physical agency that changes the state of equilibrium of the object on which it acts.

$$m\frac{d^2\vec{r}}{dt^2} = m\frac{d\vec{\mathrm{v}}}{dt} = m\vec{a} = \vec{F} = \frac{d(m\vec{\mathrm{v}})}{dt} = \frac{d\vec{p}}{dt}.$$

$$t \rightarrow -t \qquad \frac{d}{dt} \rightarrow \left(-\frac{d}{dt}\right) \qquad \dot{\vec{r}} \rightarrow -\vec{\mathrm{v}} \text{ and } \dot{\vec{\mathrm{v}}} \rightarrow \left(-\frac{\vec{F}}{m}\right)$$

Direction of velocity and acceleration both reverse.

System's trajectory would be only reversed along essentially the same path.

Newton's laws are therefore symmetric under time-reversal.

We have introduced the first two laws of Newton as fundamental principles.

Newton's III law makes a qualitative and quantitative statement about each pair of interacting objects, which exert a mechanical force on each other.

$$\vec{F}_{12} = -\vec{F}_{21}$$

In Newtonian scheme of mechanics, this is introduced as a 'fundamental' principle

–i.e., as *a law of nature.*

Newton's III law :

'Action and Reaction are Equal and Opposite'

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$\frac{d\vec{p_1}}{dt} = -\frac{d\vec{p_2}}{dt}$$

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

Newton's III Law as statement of conservation of linear momentum

We have obtained a conservation principle from 'law of nature'

Before we proceed,
we remind ourselves of another illustration of the connection between symmetry and conservation law.

Examine the ANGULAR MOMENTUM $\vec{l} = \vec{r} \times \vec{p}$
of a system subjected to a central force.

$$\vec{\tau} = \frac{d\vec{l}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \vec{r} \times \vec{F},$$

since $\dfrac{d\vec{r}}{dt} \times \vec{p} = \vec{0}$ and $\dfrac{d\vec{p}}{dt} = \vec{F}$.

$$\vec{F} = \left|\vec{F}\right| \hat{e}_r$$ SYMMETRY

$$\Rightarrow \vec{\tau} = \vec{0}$$

$\vec{l}$: constant CONSERVED QUANTITY

Coulomb's Law

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

$$\frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}^2$$

$$\cong 9.0 \times 10^9\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}^2$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$$

Maxwell's Equations

- In Maxwell's theory the speed of light, in terms of the permeability and permittivity of free space, was given by

$$v = c = 1/\sqrt{\mu_0 \varepsilon_0}$$

- Thus the velocity of light between moving systems must be a constant.

Momentum and Energy

The first term on the right-hand side is just E^2, and the second term is E_0^2. The last equation becomes

$$p^2c^2 = E^2 - E_0^2$$

We rearrange this last equation to find the result we are seeking, a relation between energy and momentum.

$$E^2 = p^2c^2 + E_0^2$$

or

$$E^2 = p^2c^2 + m^2c^4$$

Equation is a useful result to relate the total energy of a particle with its momentum. The quantities ($E^2 - p^2c^2$) and m are invariant quantities.

Coulomb's Law

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

•Central Force (acts along the line connecting the two objects)

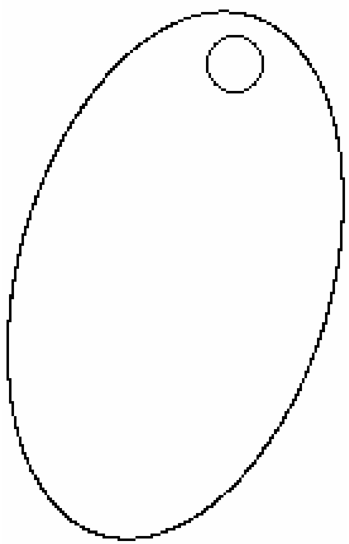
•"one over r squared force"

•Is mathematically identical to gravitational force.

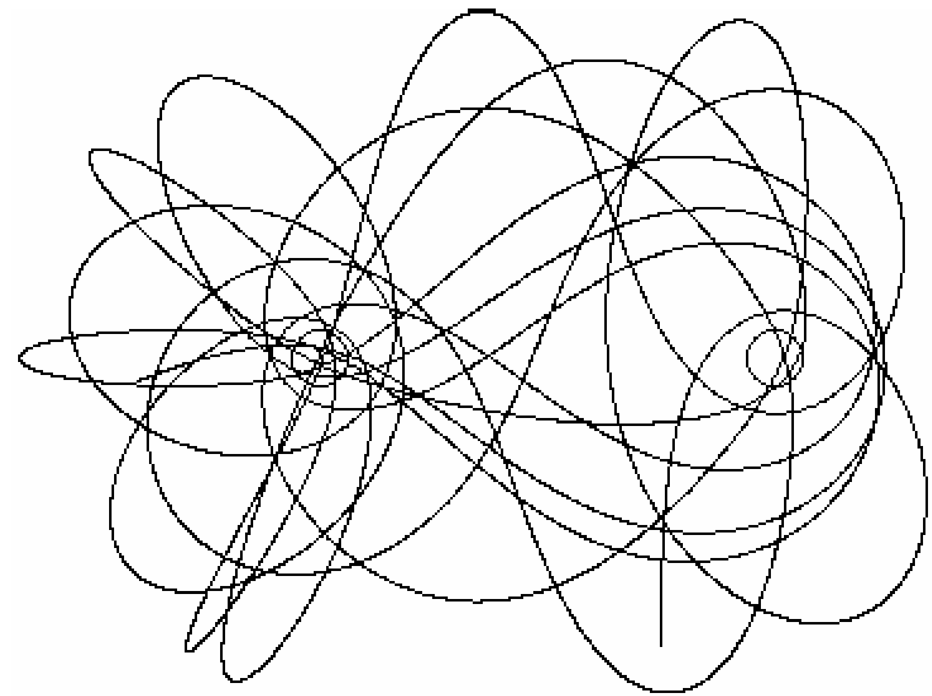
$$F = G\frac{m_1 m_2}{r^2}$$

•But they are **not** the same force.

A Planet Orbiting a Star



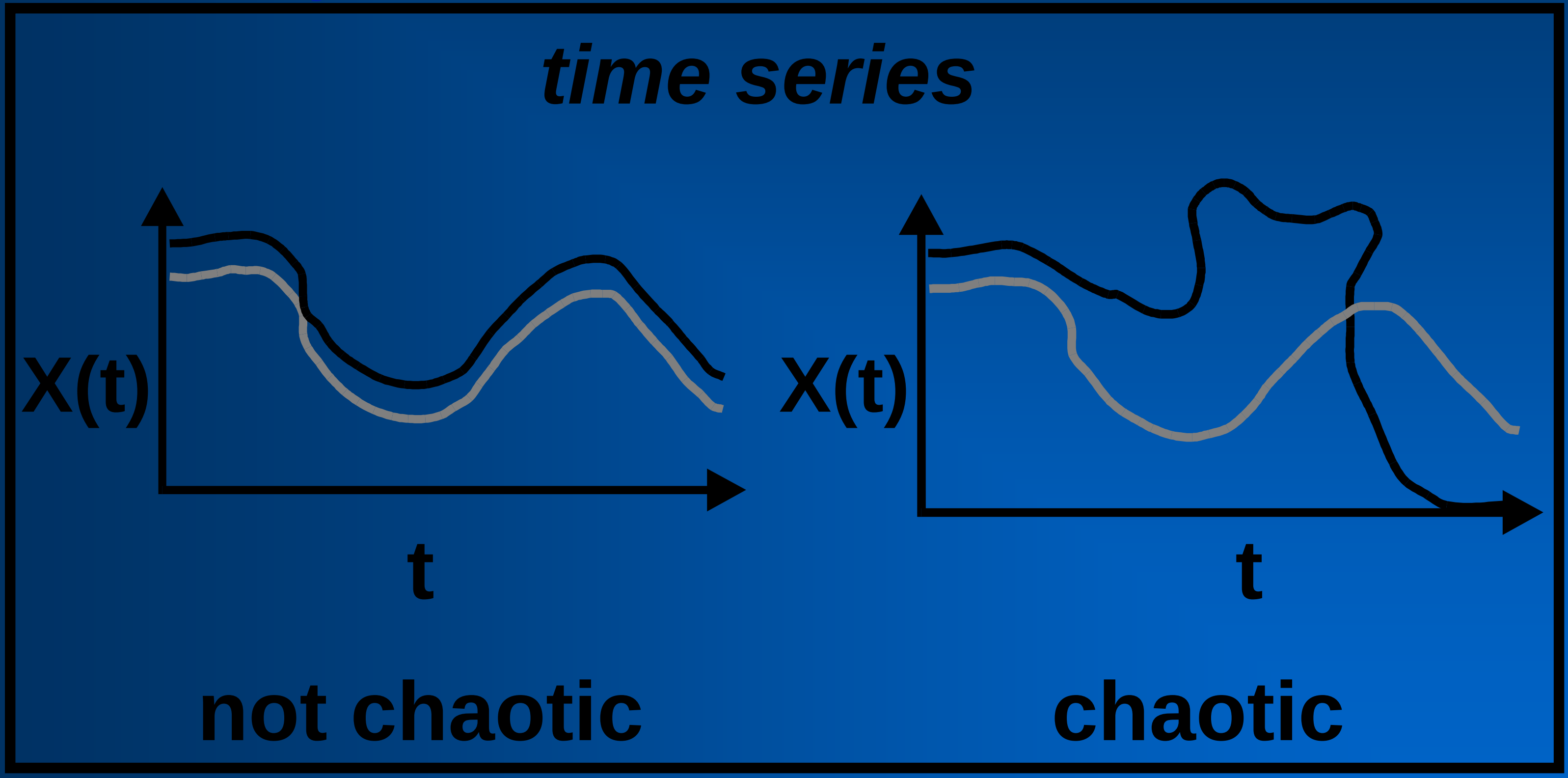
Elliptical Orbit



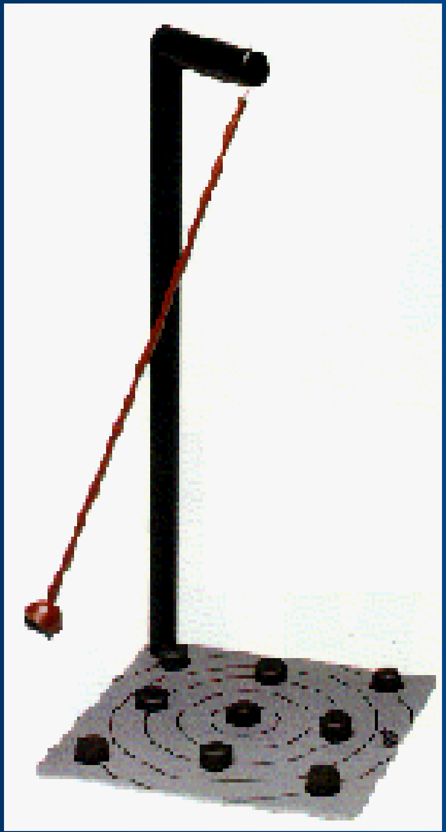
Chaotic Orbit

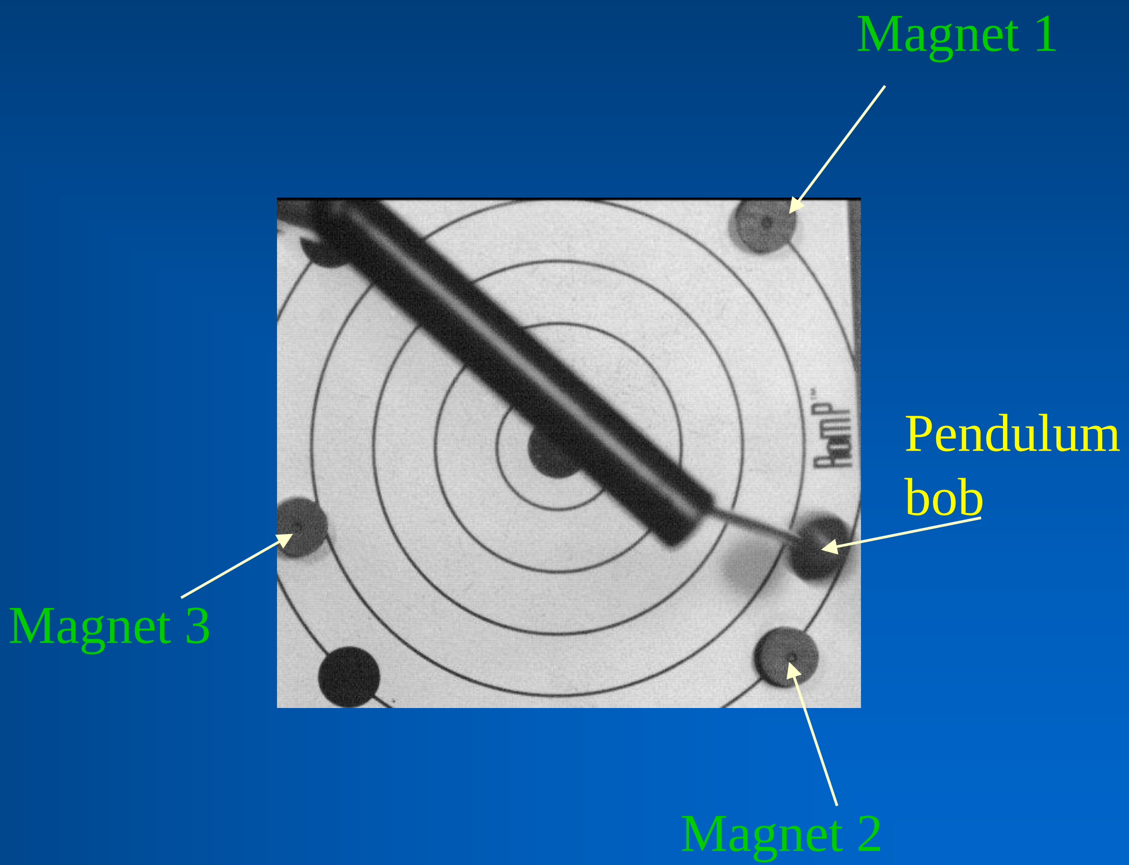
“Chaotic”

sensitivity to initial conditions



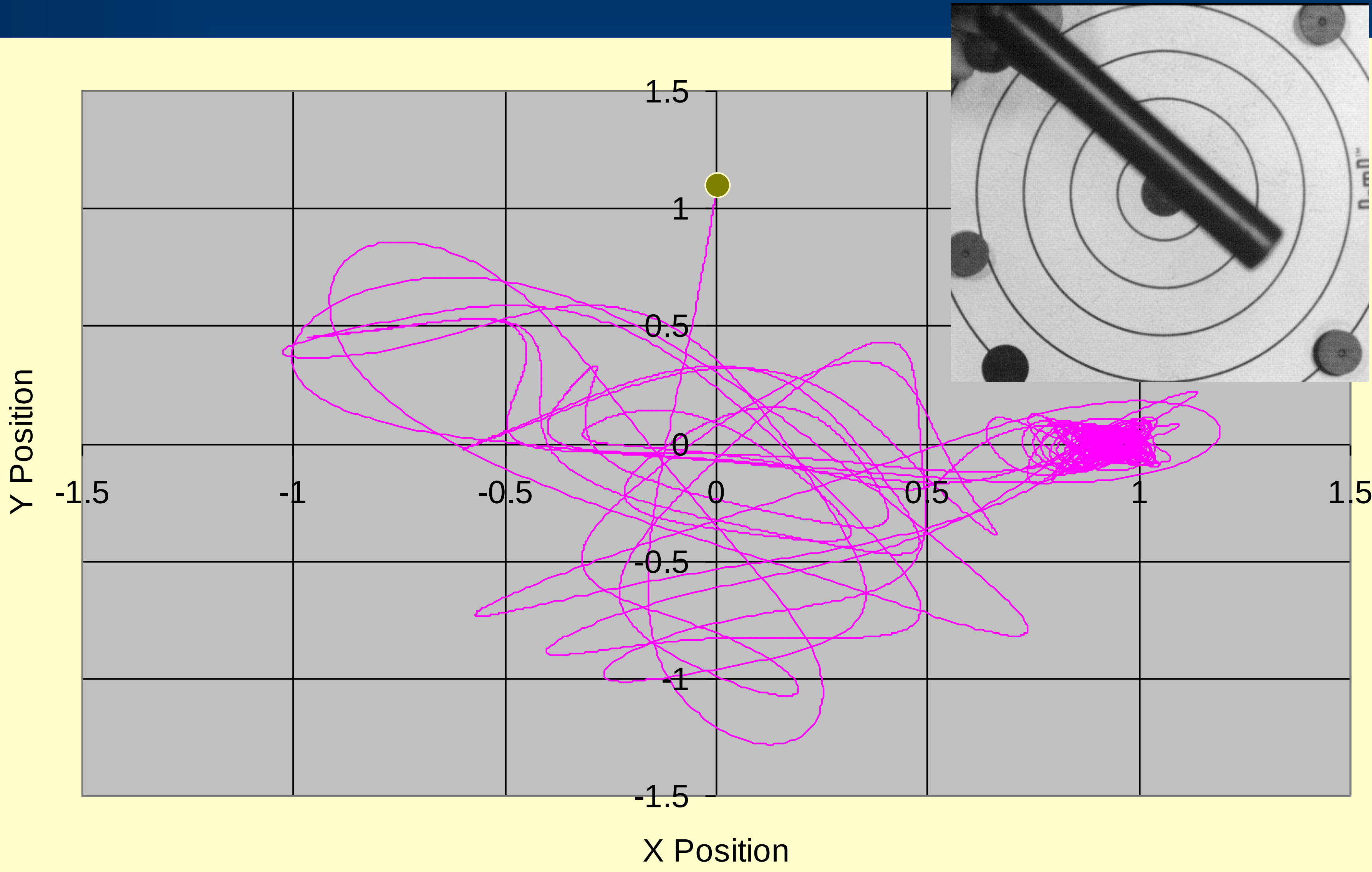
Magnetic Pendulum





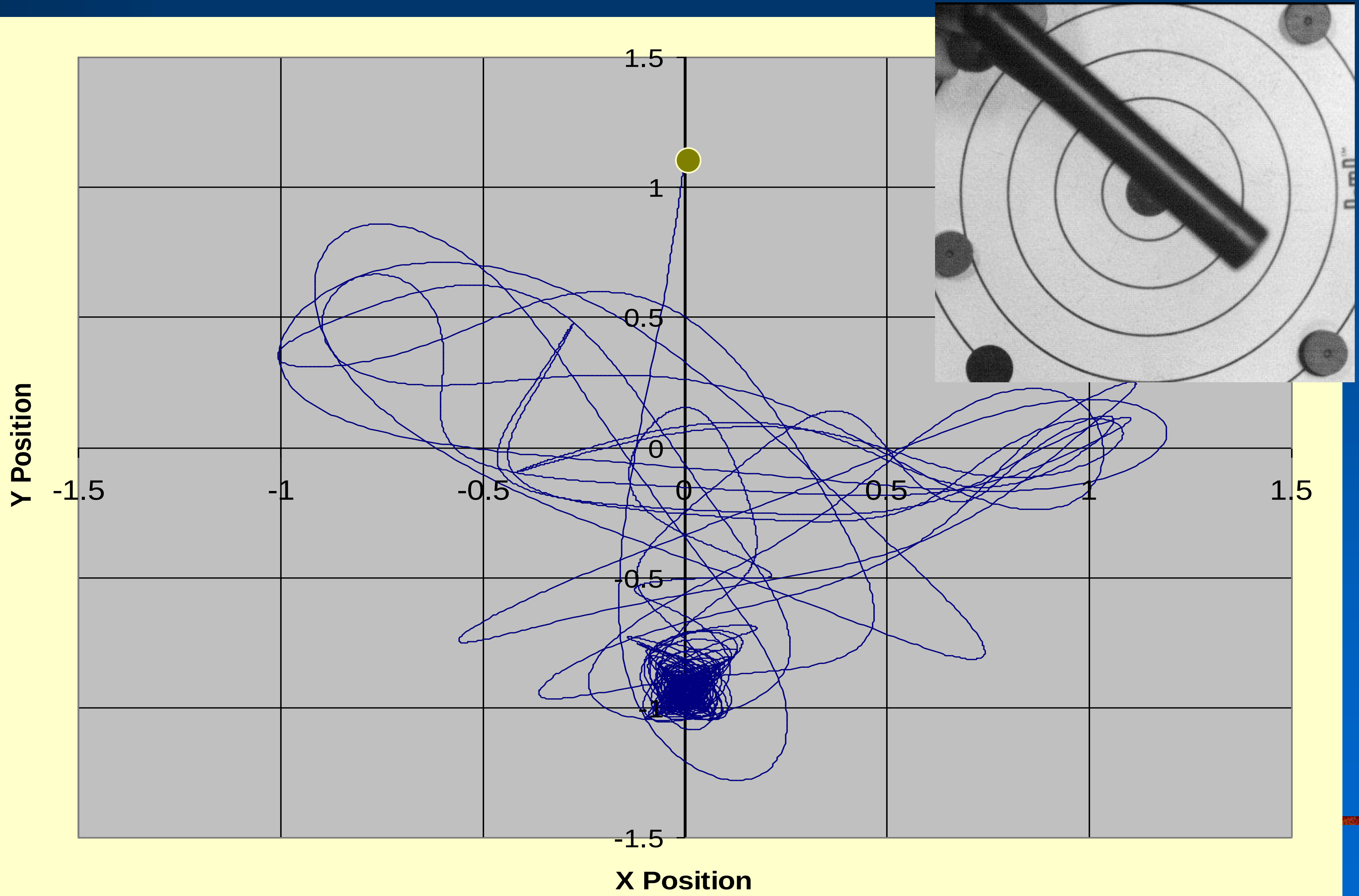
1st Trial: Trajectory of Pendulum

Numerical Solution



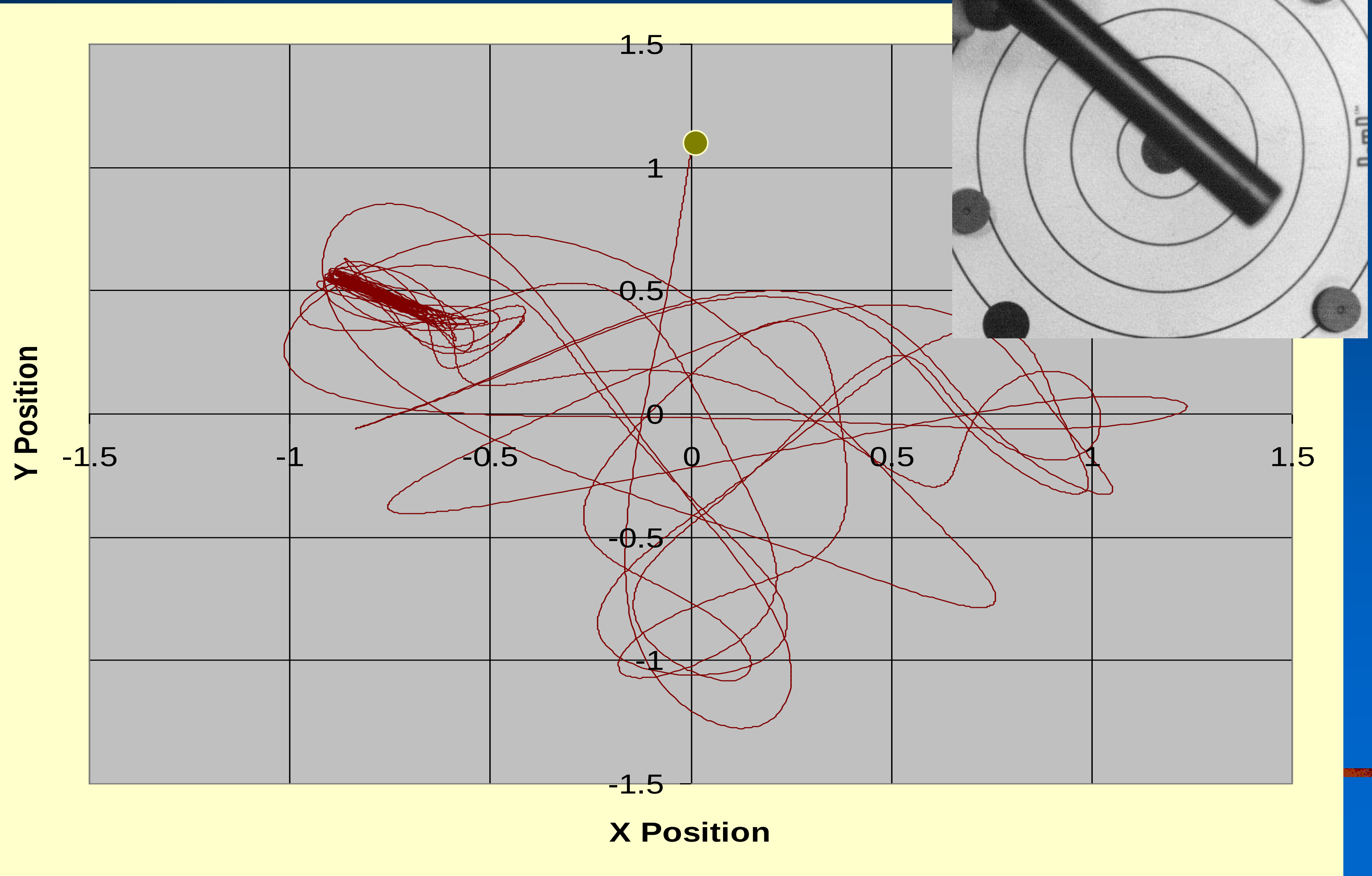
2^{nd} Trial: Trajectory of Pendulum

Numerical Solution



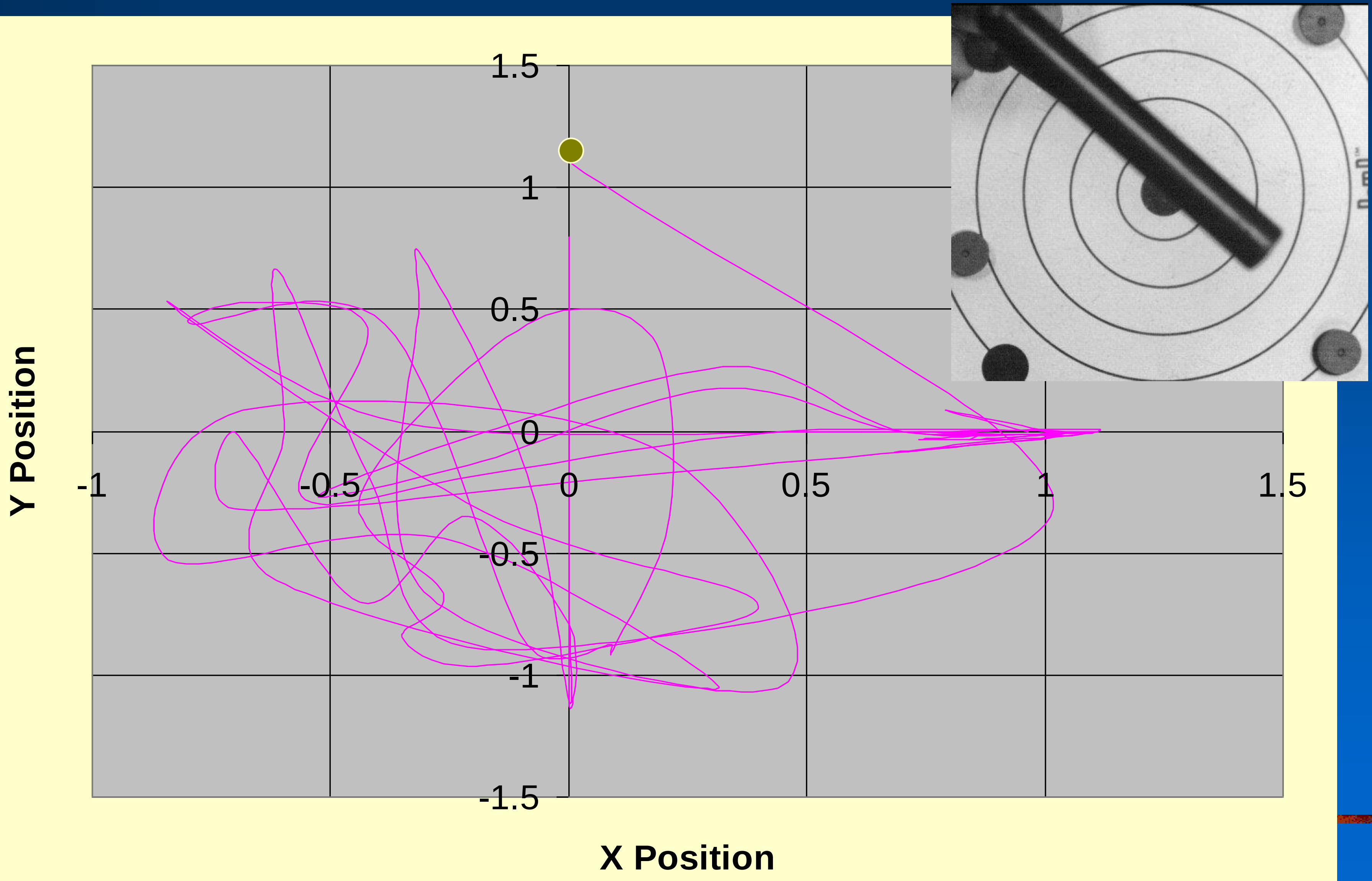
3rd Trial: Trajectory of Pendulum

Numerical Solution



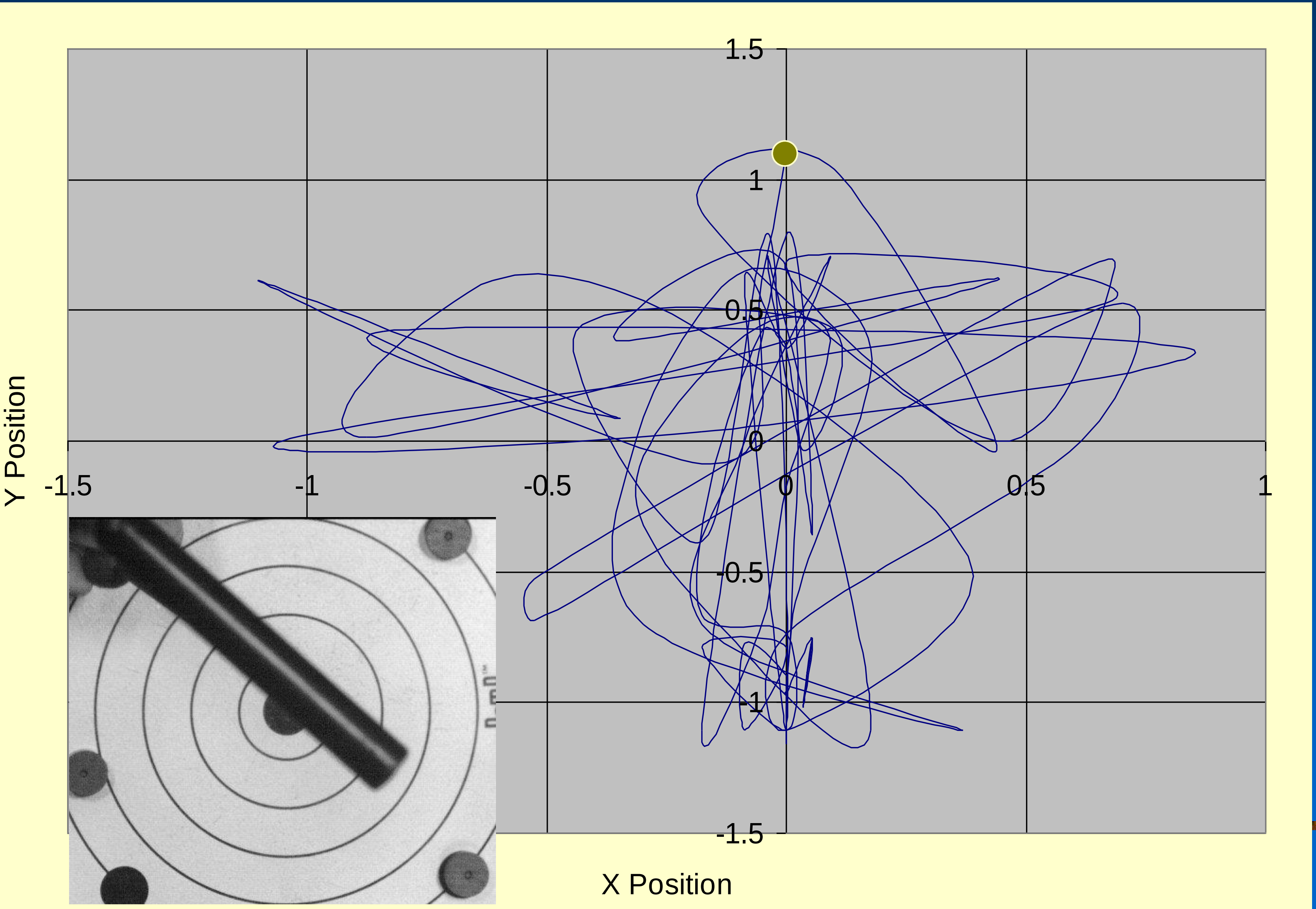
1st Trial: Trajectory of Pendulum

Experimental Solution



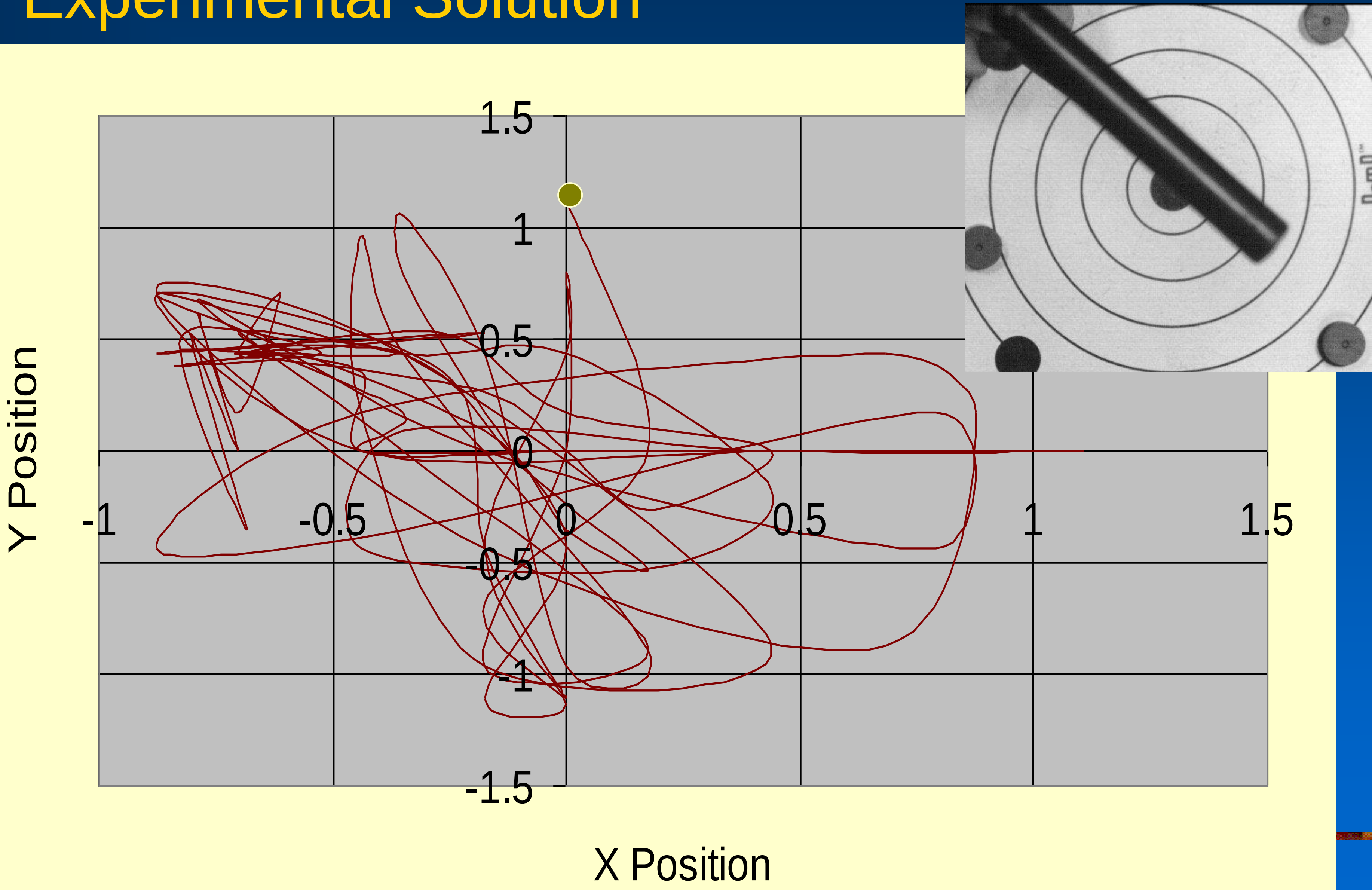
2nd Trial: Trajectory of Pendulum

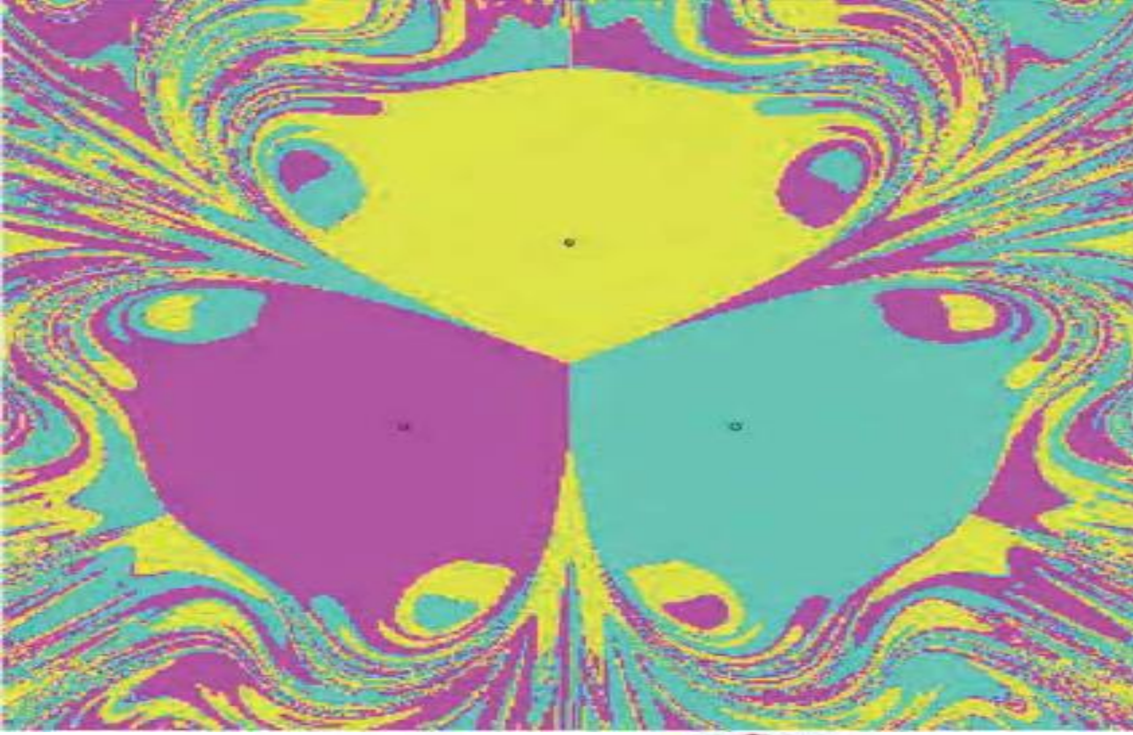
Experimental Solution

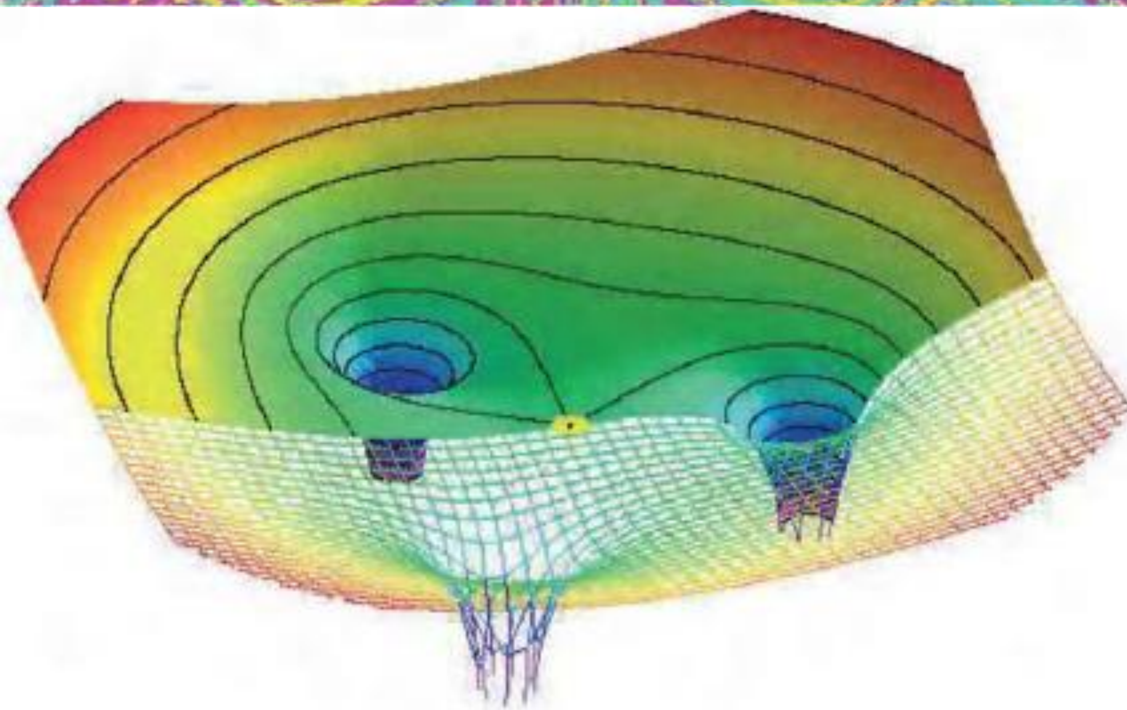


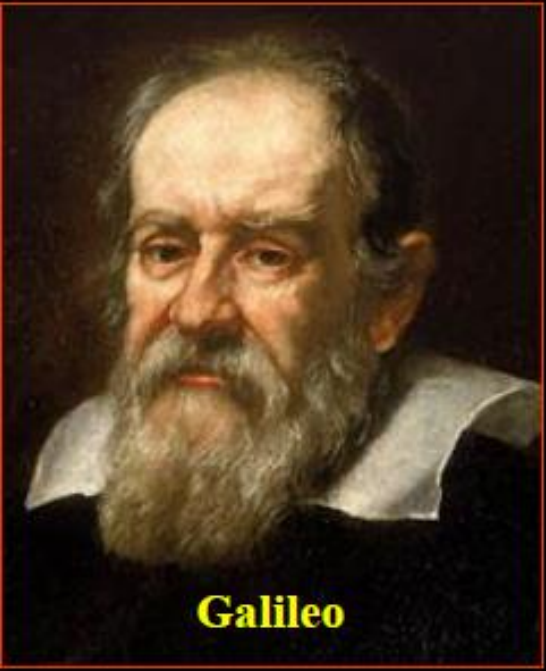
3rd Trial: Trajectory of Pendulum

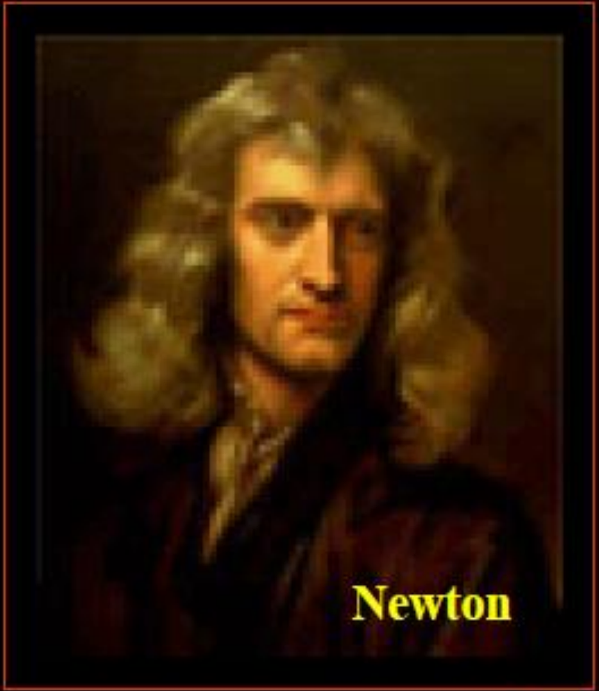
Experimental Solution











$(q, \dot{q})$

$\vec{F} = m\vec{a}$

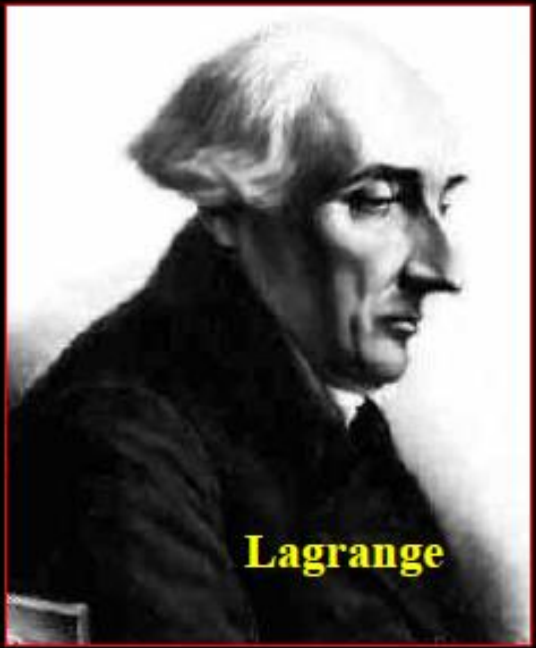
Linear Response.

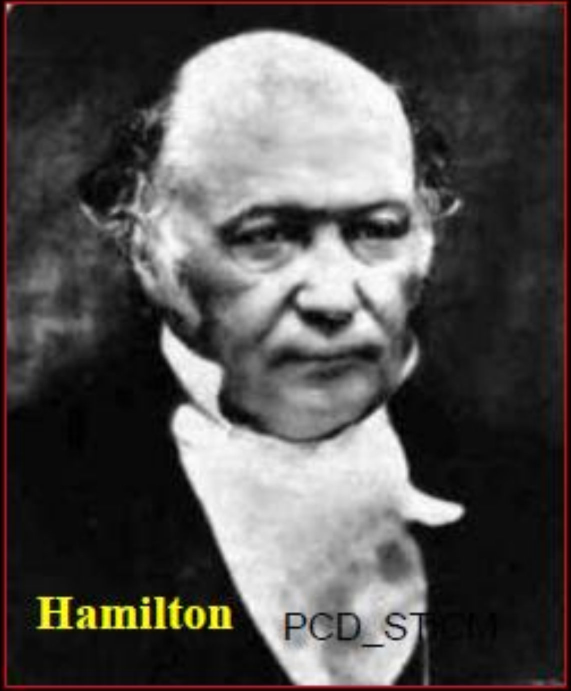
Principle of causality.

Principle of
Variation

$L(q, \dot{q})$

$H(q, p)$





$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q_k}$$

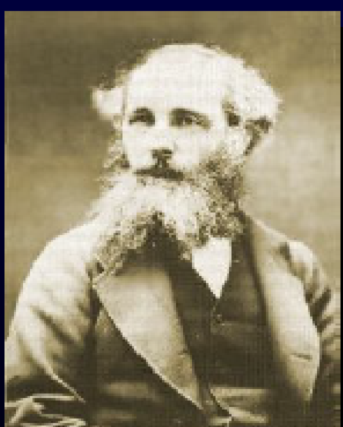
The Universe explained !

By 1900 physicists were absolutely certain of their ideas about the nature of matter and radiation. *Classical* Physics was thought to be almost complete:

Isaac Newton (1642-1727) had established

- the laws of motion
- a theory of gravitation (action at a distance)

which had stood up to stringent tests for over 200 years



James Clerk Maxwell (1831-70) developed

- a theory of electromagnetism

which not only unified electrical and magnetic fields but also explained light (and invisible radiation) in terms of electromagnetic waves.

These theories were not only mathematically rigorous but also predictive

Fundamental Assumptions of Classical Physics

Physicists at the end of the 1890s assumed the following to be <u>absolutely true</u>:

1. The universe is like a giant machine set in a framework of absolute time and space. All movement, however complicated, can be understood and explained.
2. Newtonian synthesis implies all motion has a cause - if a body is in motion it is possible to determine what causes the motion
3. If the state of motion is known at one point in space and time it can be determined at any other point in space or time.
4. The properties of light are completely described by Maxwell's electromagnetic wave theory
5. Energy can be carried either by a wave or a particle - and waves and particles are mutually exclusive
6. It is possible to measure, to any degree of accuracy, the properties of a system - whatever the system

The breakdown of Classical Physics

By 1900 three critical and reliable experiments had provided results which could not be explained by classical physics

Black body radiation ✓

The photoelectric effect

Bright line optical spectra

By 1930 an entirely new understanding of nature had been achieved - the new Quantum Physics described an alien and surprising universe.

Quantum Physics together with Relativity had overturned Classical Physics entirely

The Photoelectric Effect

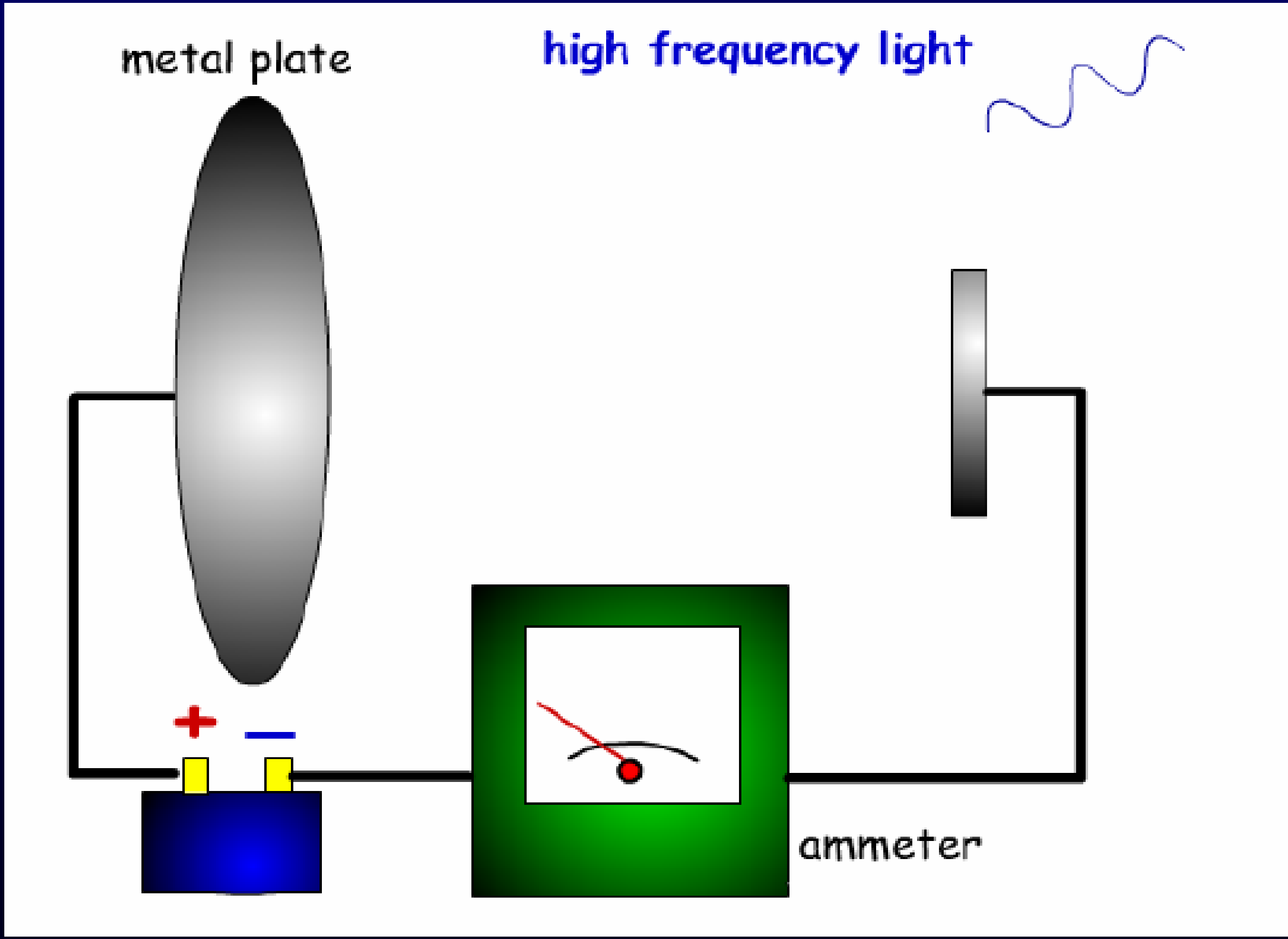
first discovered by Heinrich Hertz in 1887

The photoelectric effect

Philipp Lenard (1862-1947) carried out a series of careful experiments on the photoelectric effect between 1899 and 1902

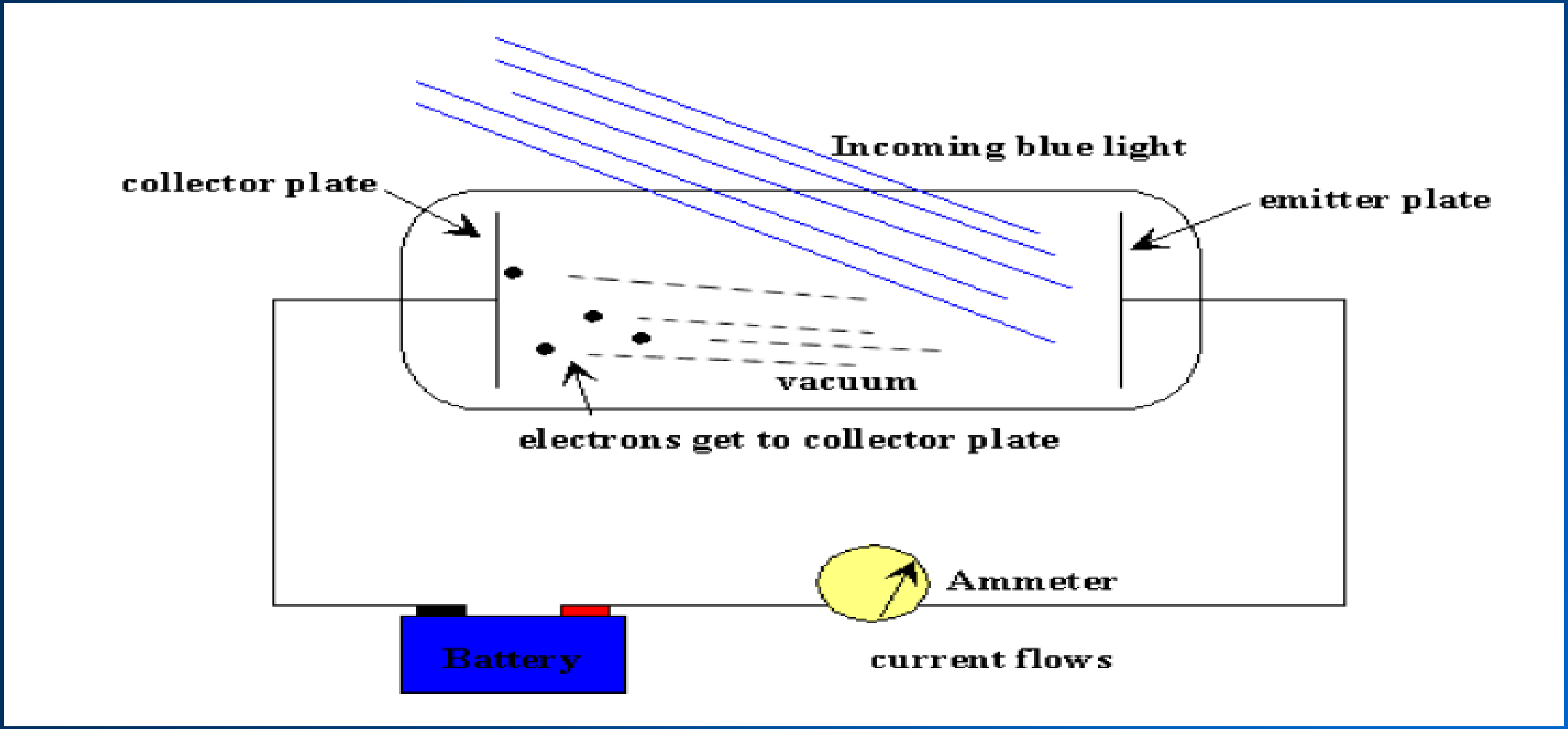
Monochromatic (ie single frequency) radiation was shone on a variety of metals and the number and energy of emitted electrons were measured

The photoelectric effect

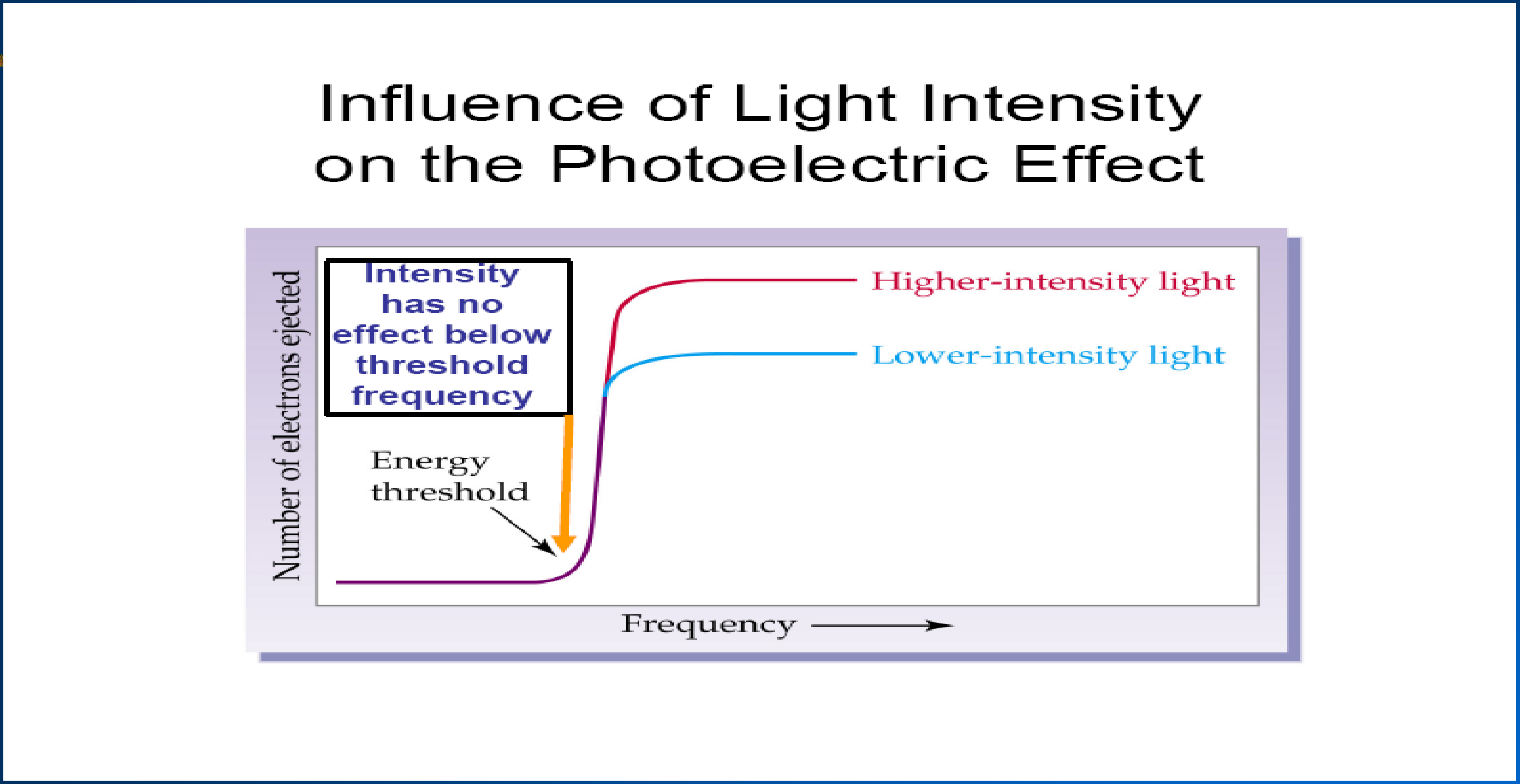


Basics of Quantum Mechanics
- Photoelectric Effect -

A Photocell is Used to Study the Photoelectric Effect

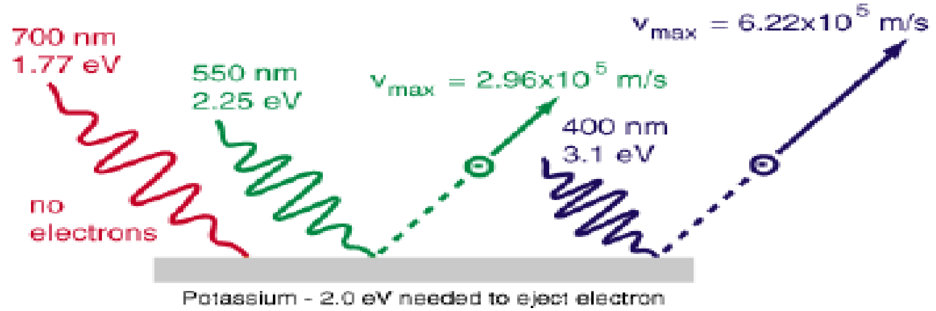


Basics of Quantum Mechanics
- Photoelectric Effect -



Larger light intensity means larger number of photons at a given frequency (Energy)

Basics of Quantum Mechanics
- Photoelectric Effect -



Photoelectric effect

Light can eject electrons from a metal, but only if its frequency is above a threshold frequency (characteristic for each metal).

Classically, for light as a wave, its energy is proportional to the square of its *amplitude*.

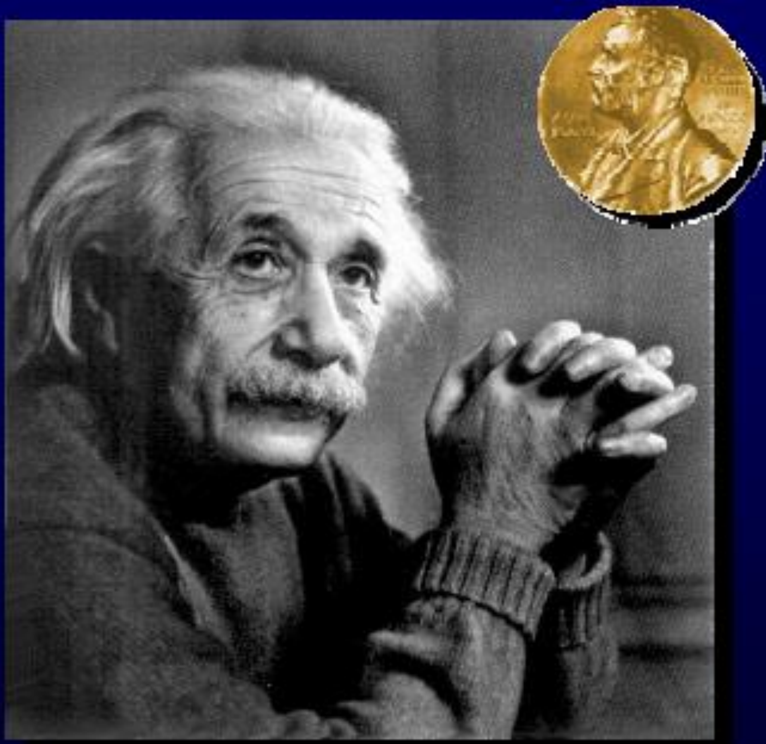
For particles, energy is proportional to *frequency*

Einstein (1905) proposed that light has particle nature (as well as wave nature).

light is quantized (photons).

Larger frequency, means smaller wavelength, and larger Energy=hf.

Albert Einstein



In 1905 in the same volume of *Annalen der Physik* Albert Einstein published three theoretical papers that revolutionized physics. In these papers he:

Introduced the Theory of Relativity

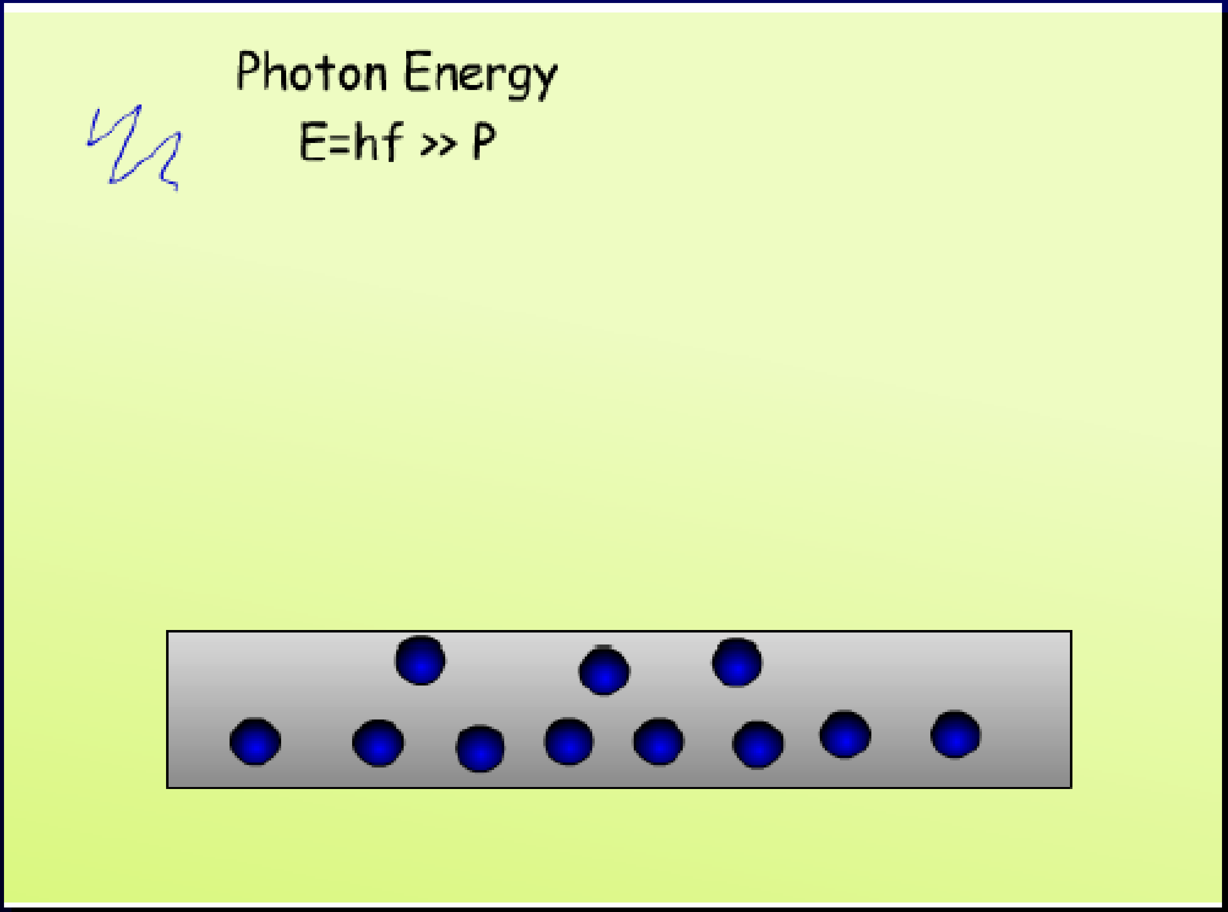
Proved the existence of atoms

Explained the photoelectric effect

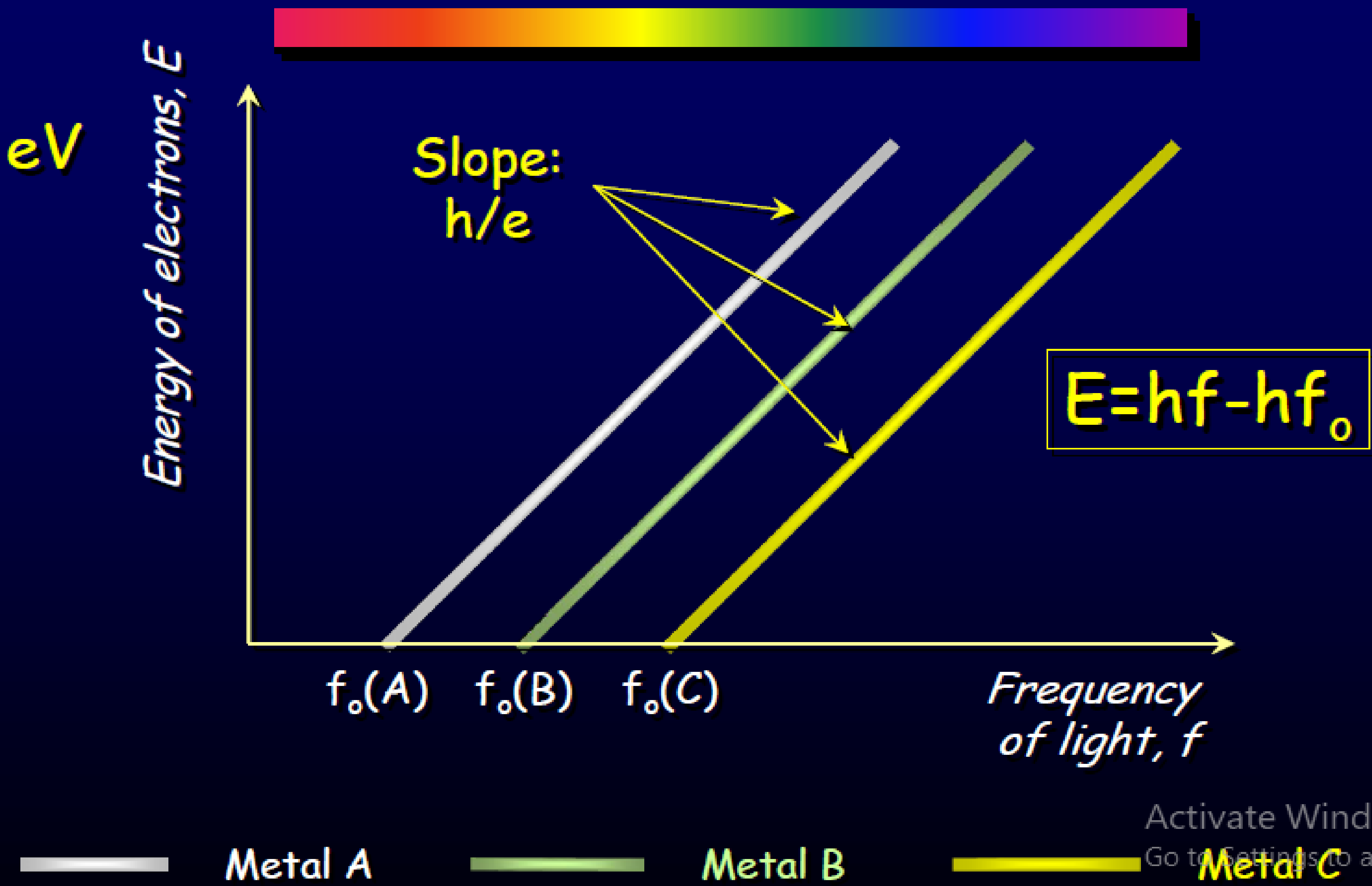
It was this latter work, which introduced an heuristic theory on the quantum nature of light, for which he was awarded the Nobel Prize in 1921

(<u>not</u> for the Theory of Relativity!)

Einstein's model



A summary of the photoelectric effect



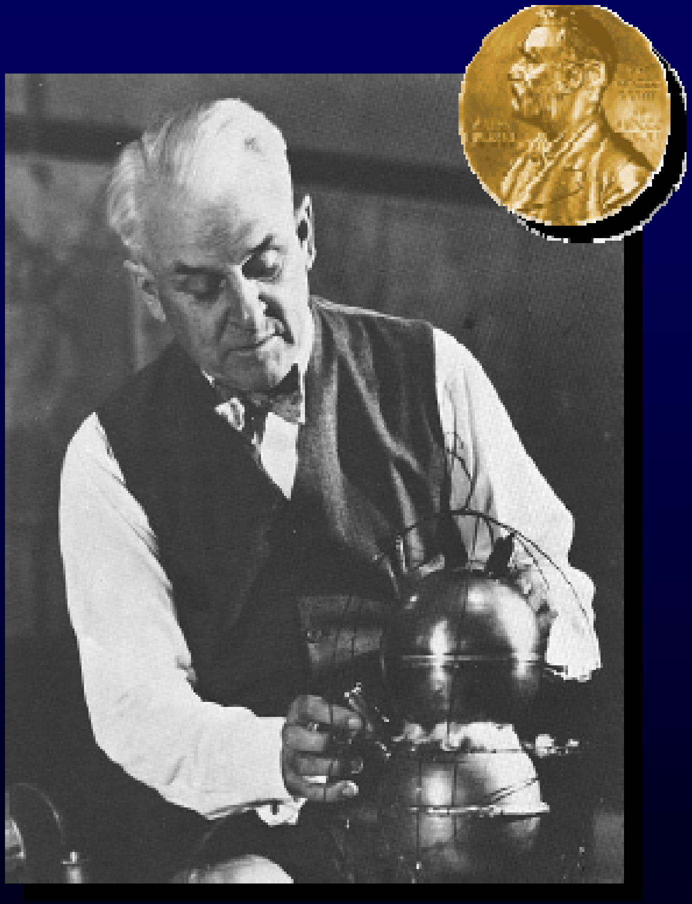
Basics of Quantum Mechanics - Photoelectric Effect -

- **The photoelectric effect provides evidence for the particle nature of light.**
- **It also provides evidence for quantization.**
- **If light shines on the surface of a metal, there is a point at which electrons are ejected from the metal.**
- **The electrons will only be ejected once the threshold frequency is reached .**
- **Below the threshold frequency, no electrons are ejected.**
- **Above the threshold frequency, the number of electrons ejected depend on the intensity of the light.**

Classical Physicists vs Quantum Physicists

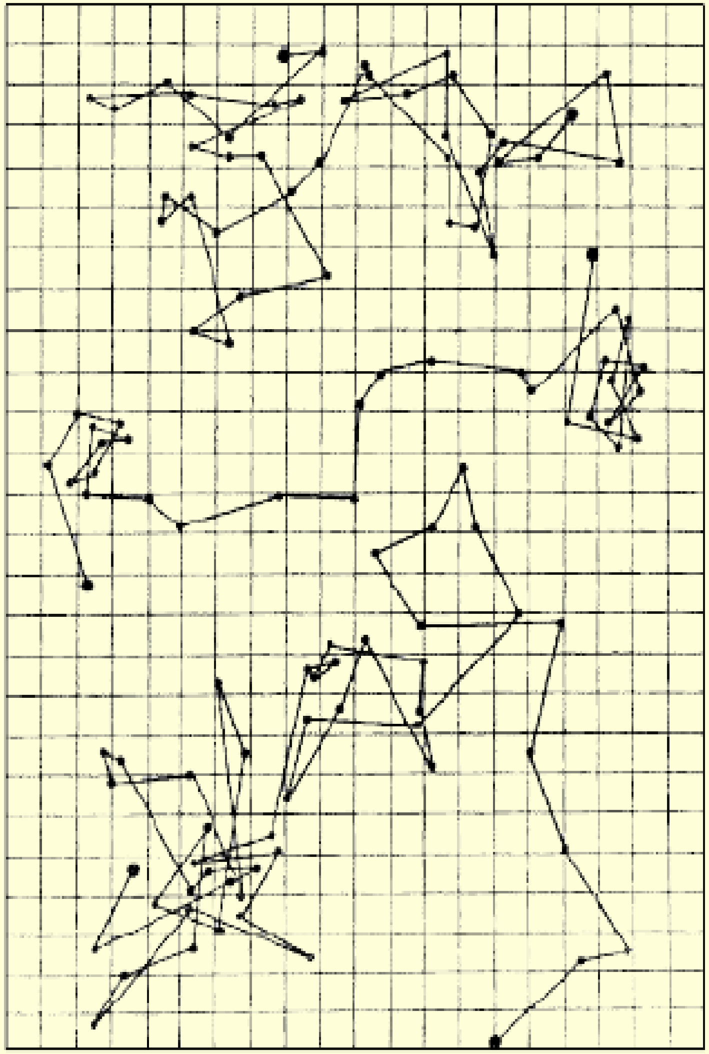
For many years the ideas of quantised energy and quantised radiation were not well accepted - even Planck rejected both his own and Einstein's ideas

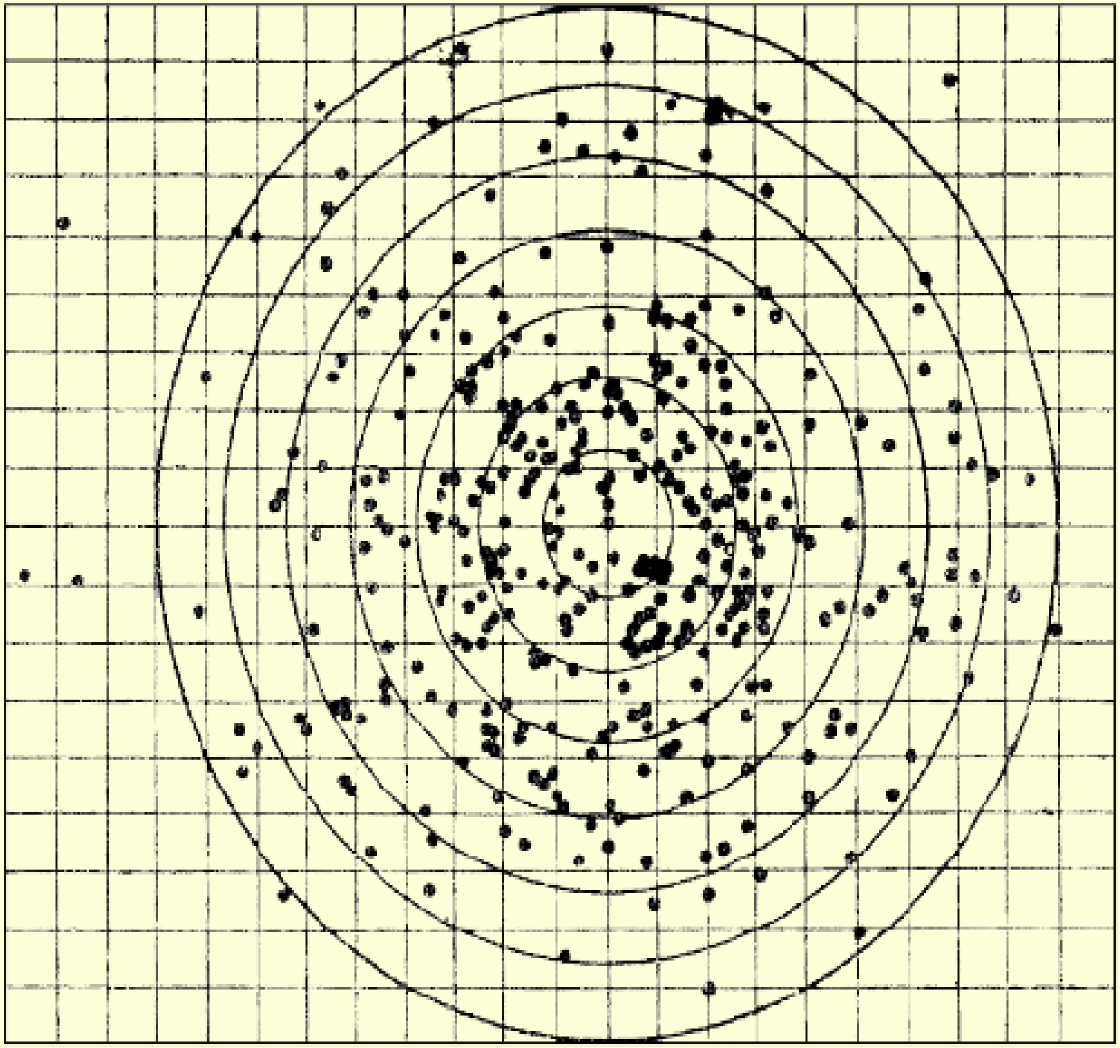
Robert Millikan (1868-1953)
devoted almost ten years to trying to disprove Einstein's explanation of the photoelectric effect



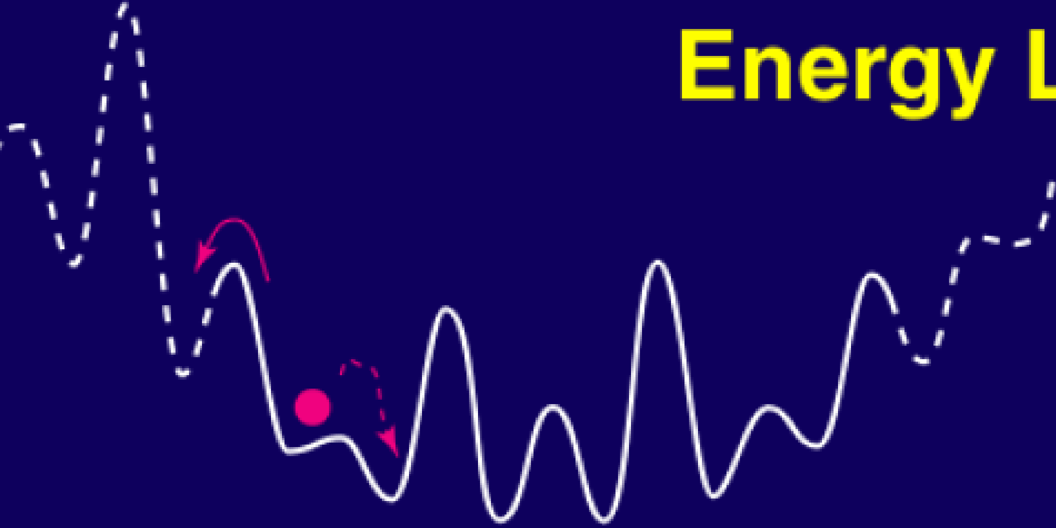
He produced extremely accurate data which instead strongly supported the Einstein model of quantised radiation - and in 1923 was awarded the Nobel Prize for his efforts

Brownian Motion



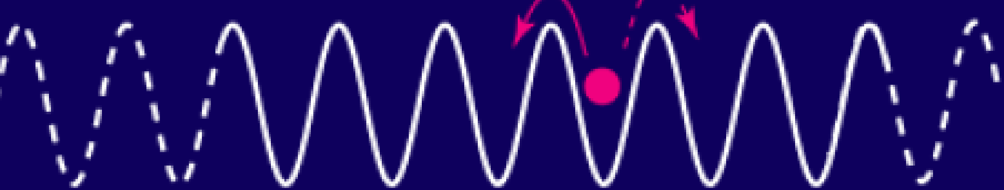


Energy Landscape Picture

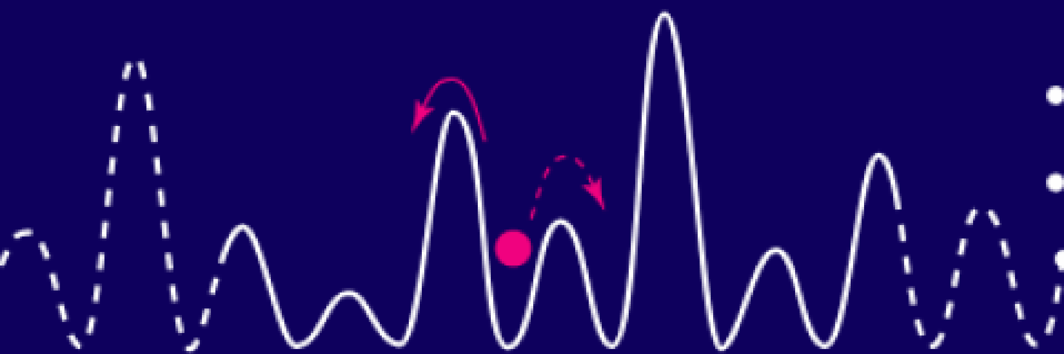


- A distribution of confromation
- Barrier heights unknown

Brownian Diffusion



- Equal barrier heights
- $\langle(\Delta x)^2\rangle \propto t$
- The distribution of waiting time in each well is exponential.



- A distribution of barrier heights
- Subdiffusion $\langle(\Delta x)^2\rangle \propto t^{\alpha}$, $0<\alpha<1$
- The waiting time between "hops" follows power law distribution, $w(t) \sim \tau^{\alpha} / t^{1+\alpha}$.

no finite first moment

Anomalous Diffusion

The Diffusion Equation

A. Fick

M. Smoluchowski

A. Einstein

$$\frac{\partial}{\partial t} P(x,t) = K \frac{\partial^2}{\partial x^2} P(x,t)$$

The Diffusion Equation (1855)

Continuity

$$\frac{\partial}{\partial t} n(\vec{x},t) = -\mathrm{div}\,\vec{j}(\vec{x},t)$$

(1914,1915,1918)

+ linear response

$$\vec{j}(\vec{x},t) = -K\,\mathrm{grad}\,n(\vec{x},t) \quad + \mu n(\vec{x},t)\vec{f}(\vec{x},t)$$

=> the diffusion equation

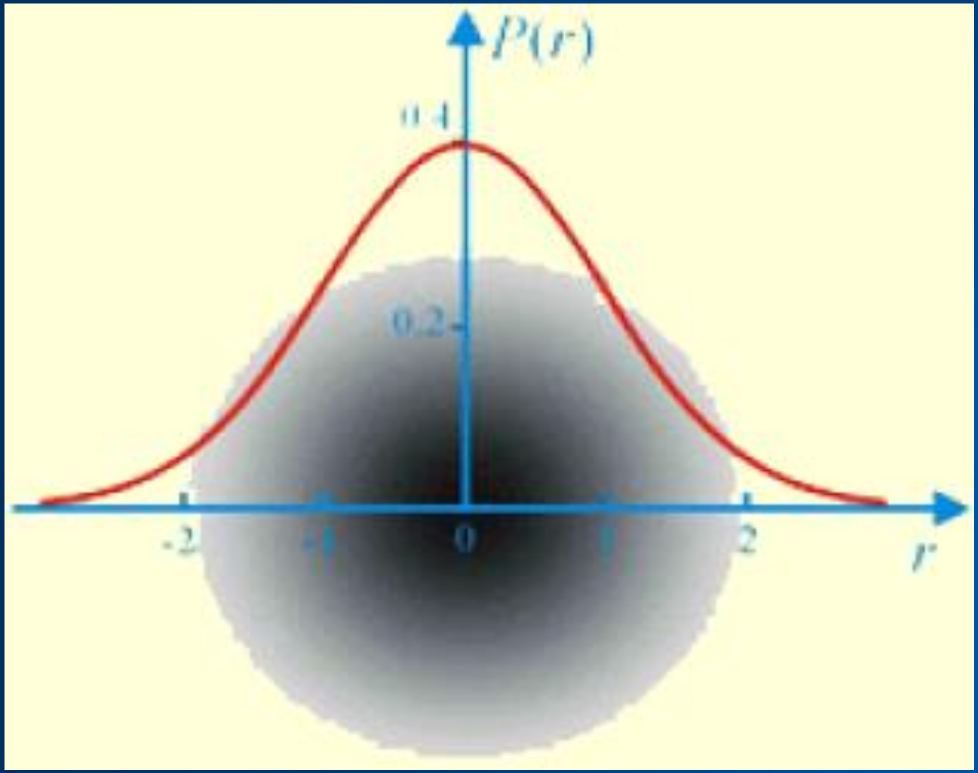
$$\frac{\partial}{\partial t} n(\vec{x},t) = K\Delta n(\vec{x},t) \quad -\nabla\left(\mu\vec{f}(\vec{x},t)n(\vec{x},t)\right)$$

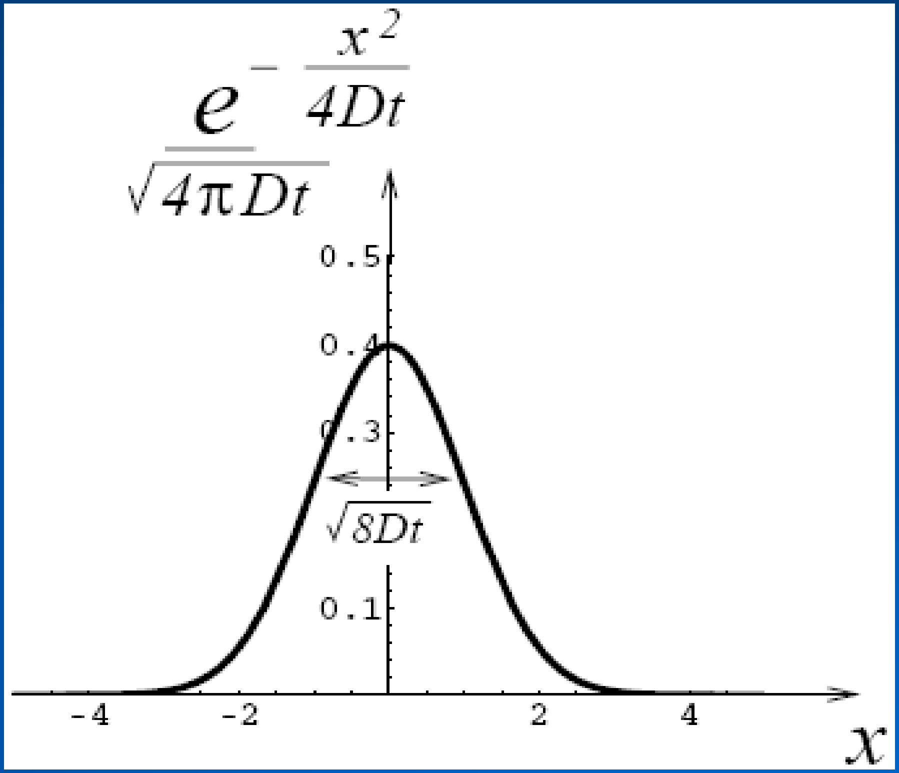
the Green's function solution

$$n(\vec{x},t) = (4\pi K t)^{-d/2} \exp\left(-\frac{\vec{x}^2}{4Kt}\right)$$

Essentially an equation for the pdf: $n(\vec{x},t) \rightarrow P(\vec{x},t)$

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}, \quad \langle x^2 \rangle_t = \int_{-\infty}^{+\infty} x^2 P(x,t)\,\mathrm{d}x = 2Dt$$





Basic properties of the wave equation

The wave equation (WE) writes:

$$u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty.$$

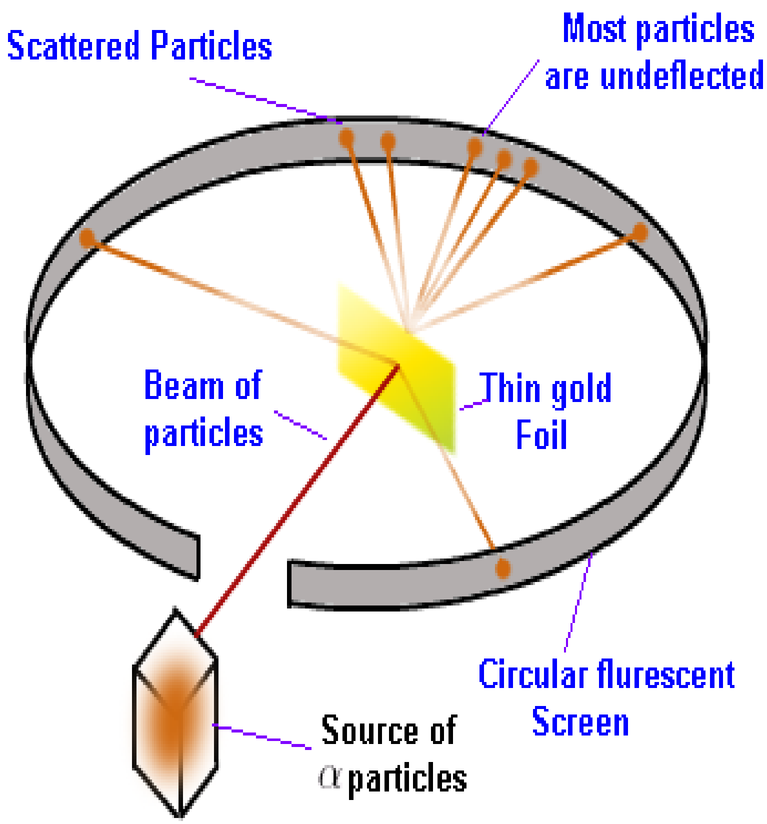
where the following notation is used for the derivatives: $\partial u / \partial x = u_x$ ...

The WE has the following basic properties:

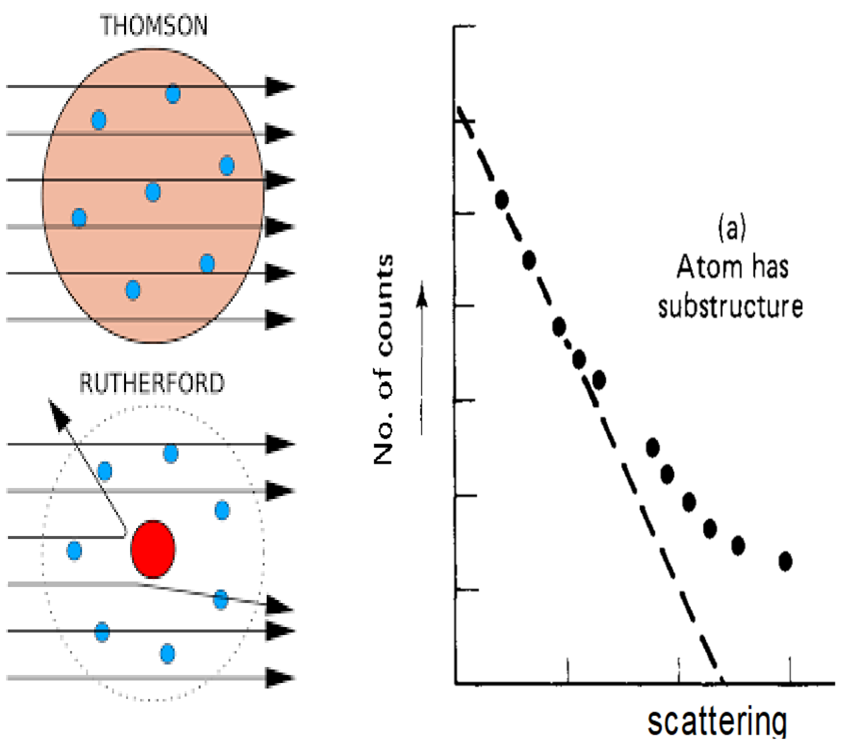
- it has two independent variables, x and t, and one dependent variable u (i.e. u is an unknown function of x and t);
- it is a *second*-order PDE, since the highest derivative in the equation is second order;
- it is a homogeneous linear PDE.

Rutherford Scattering

Griffiths §1.1

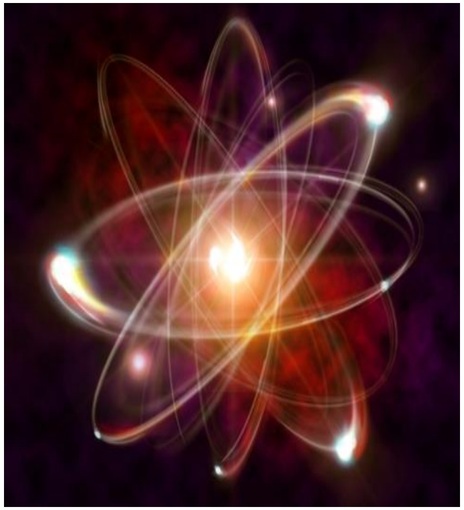


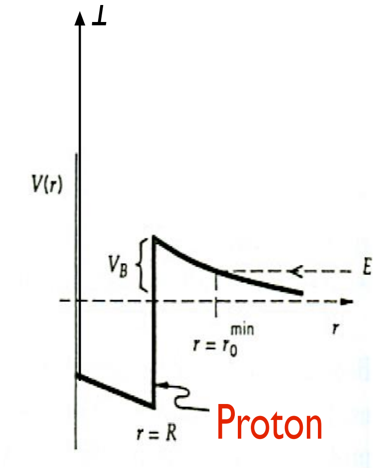
- Rutherford (1911)
  - α-particles at gold target
  - Most particle pass undisturbed, few have a "hard" collision
  - Atom has substructure: small heavy nucleus

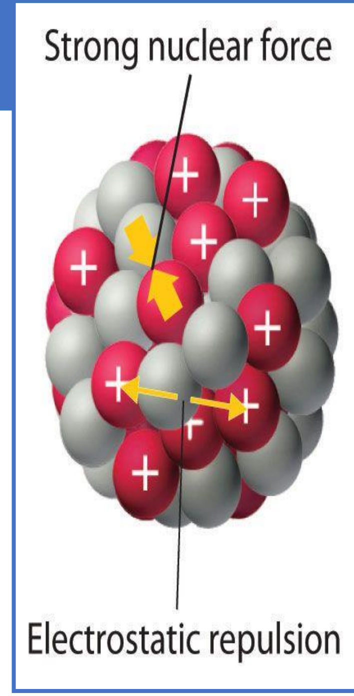


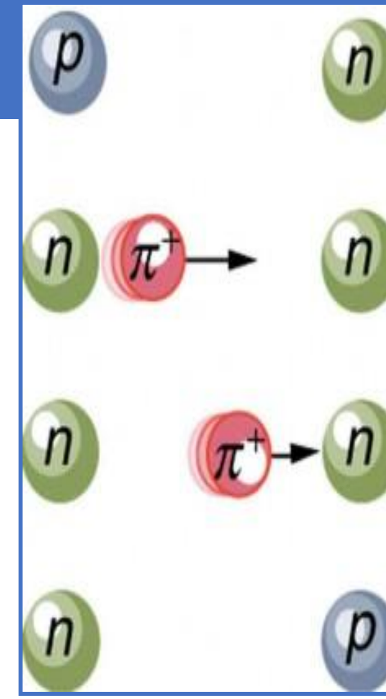
- Atom is basically just vacuum:
  - Atom size: ~ $10^{-10}\ m$
  - Nucleus size: ~ $10^{-15}\ m$

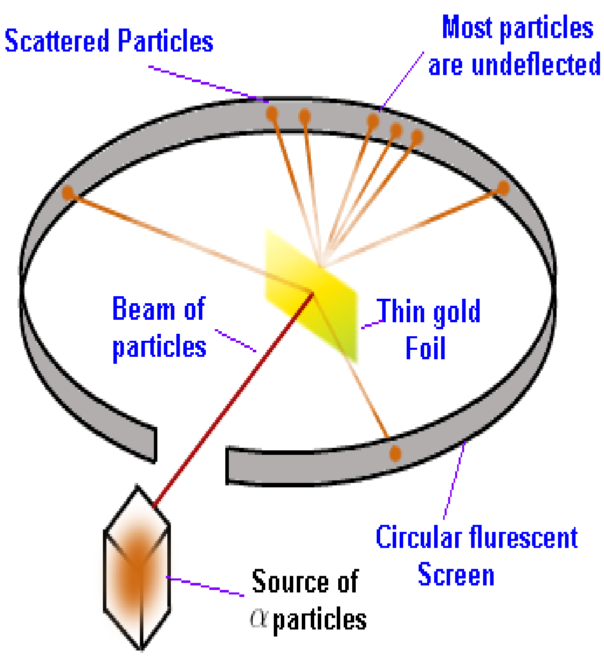
“Particles” - Nuclear

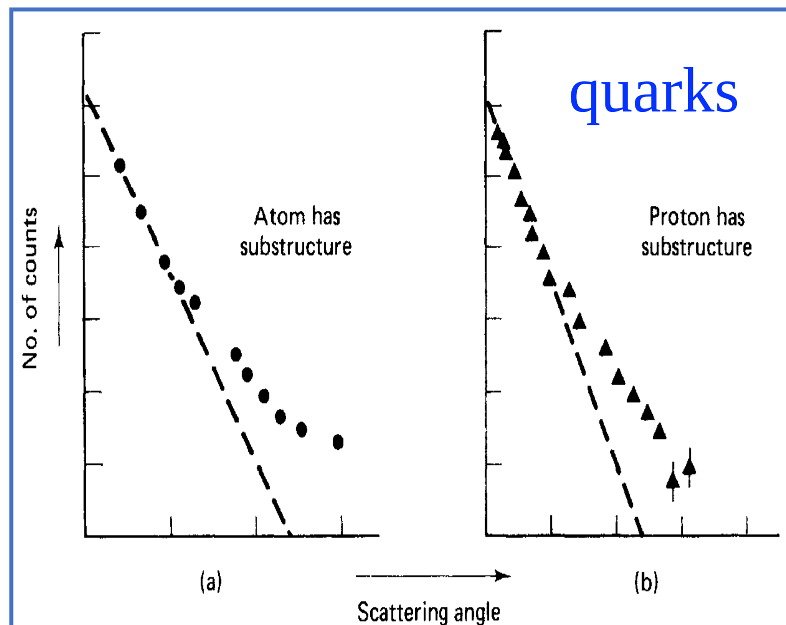






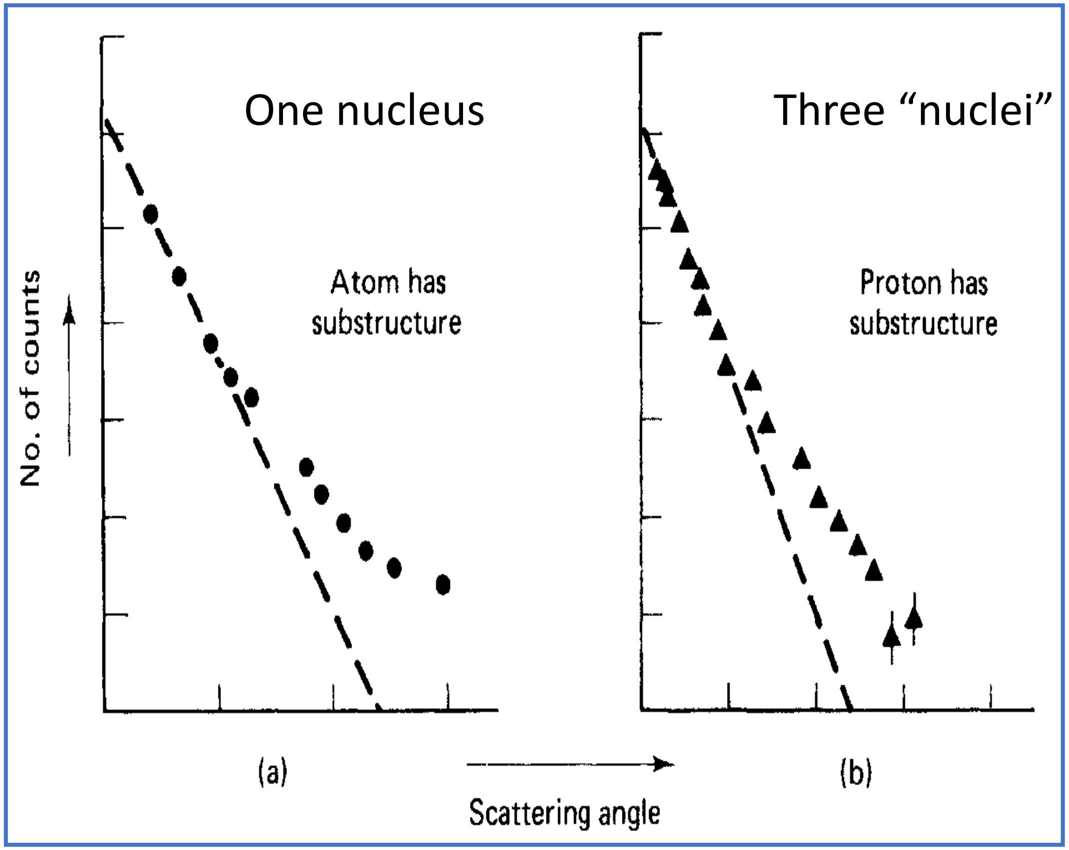






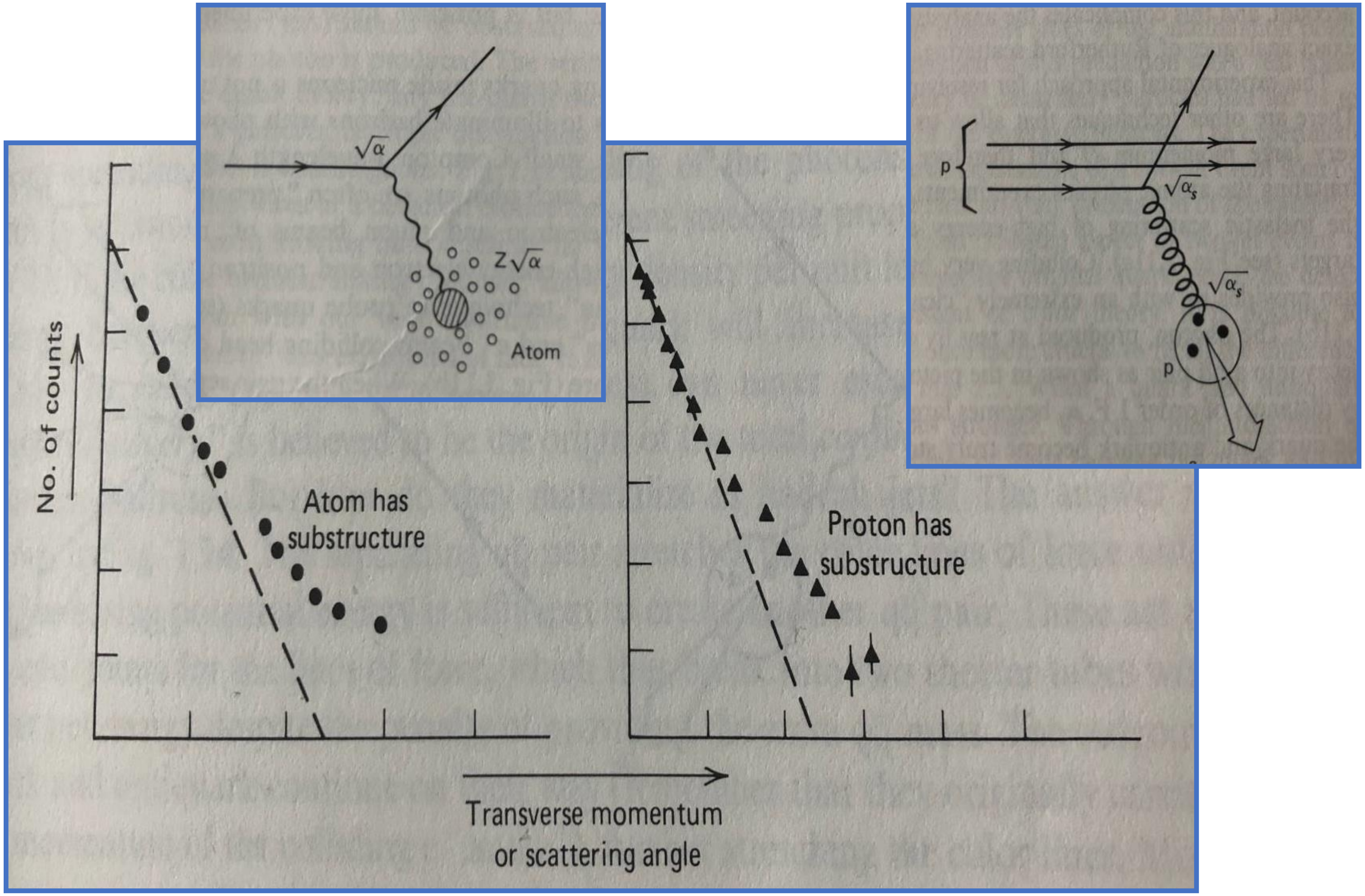
Deep Inelastic Scattering ("DIS")

- Indeed individual quarks have never been observed.
  - Do they actually exist or are they just a mathematical bookkeeping tool?
- ~1970 SLAC: "DIS"
  - Proton substructure is seen similar to Rutherford scattering
- Quarks are real and they carry an additional quantum number which can have three values
  - *"Color" is just a name*



Proton Substructure: discovery of “partons” or “quarks”(1968)

- Similarly to Rutherford scattering (“substructure of the nucleus”) deep inelastic scattering of electrons on protons show the proton substructure

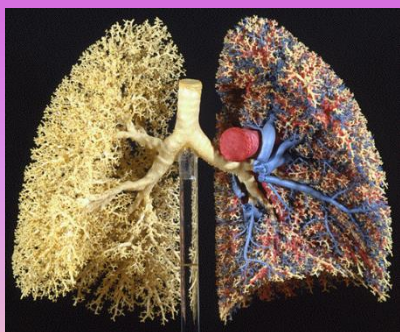


Guess the Mathematics behind this…..

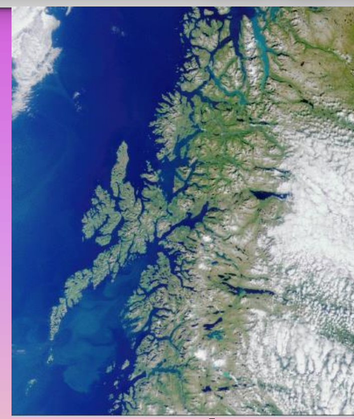
LIFE is Complex but … Simple

Zahran

Examples of natural (random) fractals



Lung

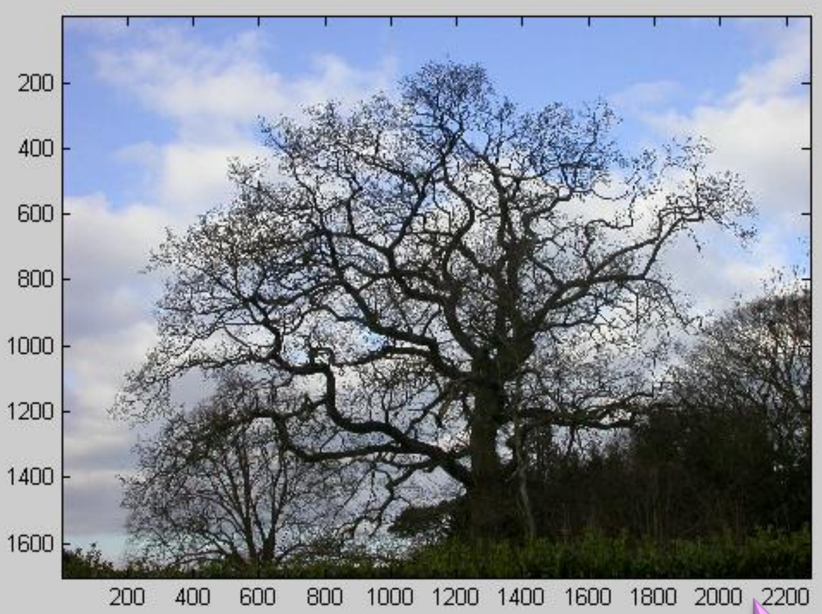


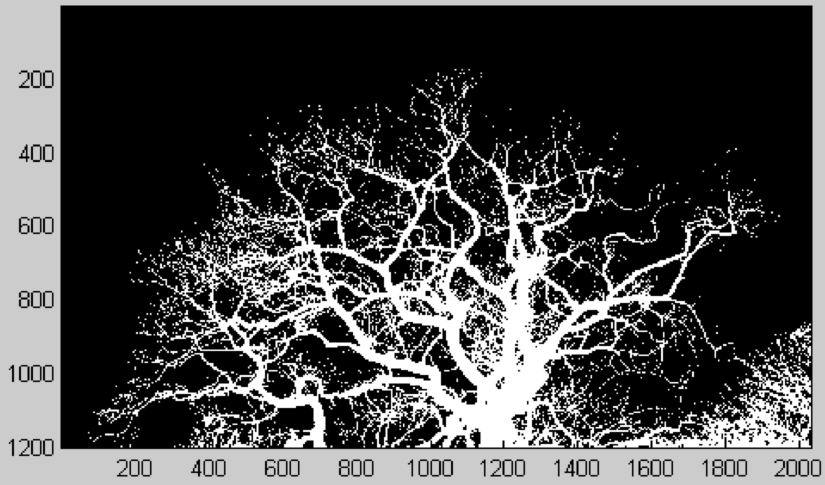
Geographic map

mountain

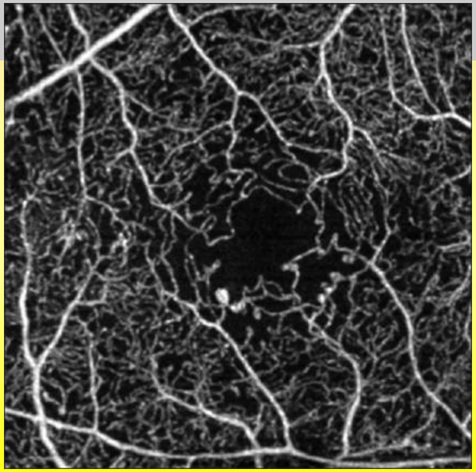
Vessels

Fractal geometry





So the retinal blood vessel structure branching pattern resembles fractal



Branching Patterns

nerve cells

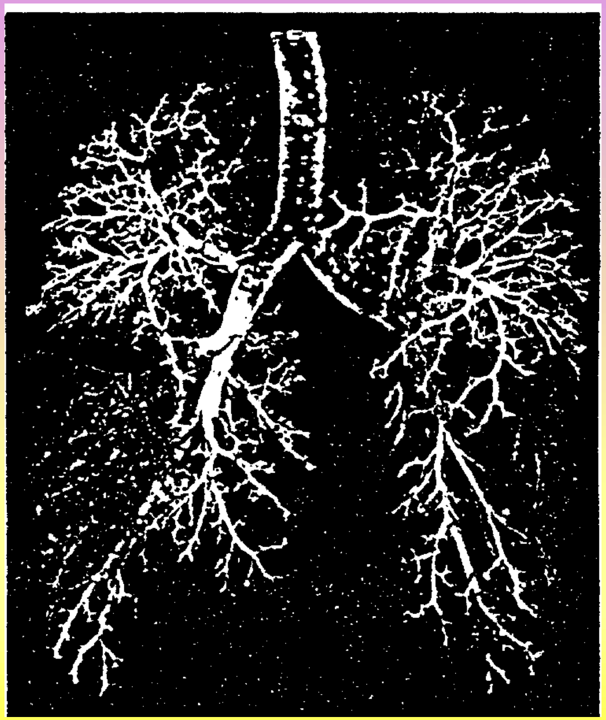
in the retina, and in culture

Caserta, Stanley, Eldred, Daccord, Hausman, and Nittmann 1990
Phys. Rev. Lett. 64:95-98
Smith Jr., Marks, Lange, Sheriff Jr., and Neale 1989
J. Neurosci, Meth. 27:173-180

Branching Patterns

blood vessels
in the retina
Family, Masters, and Platt 1989
Physica D38:98-103
Mainster 1990 Eye 4:235-241

air ways
in the lungs
West and Goldberger 1987
Am. Sci. 75:354-365



Brief history about fractal geometry

Fractals: theory and applications

Benoît Mandelbrot (1924-2010)

- **A fractal is generally**
  "a rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole"

- This property is called self-similarity.

(B. Mandelbrot)

- "The Geometry of any real object in nature is not regular such as clouds are not sphere, mountains are not cones, and coastlines are not circles."

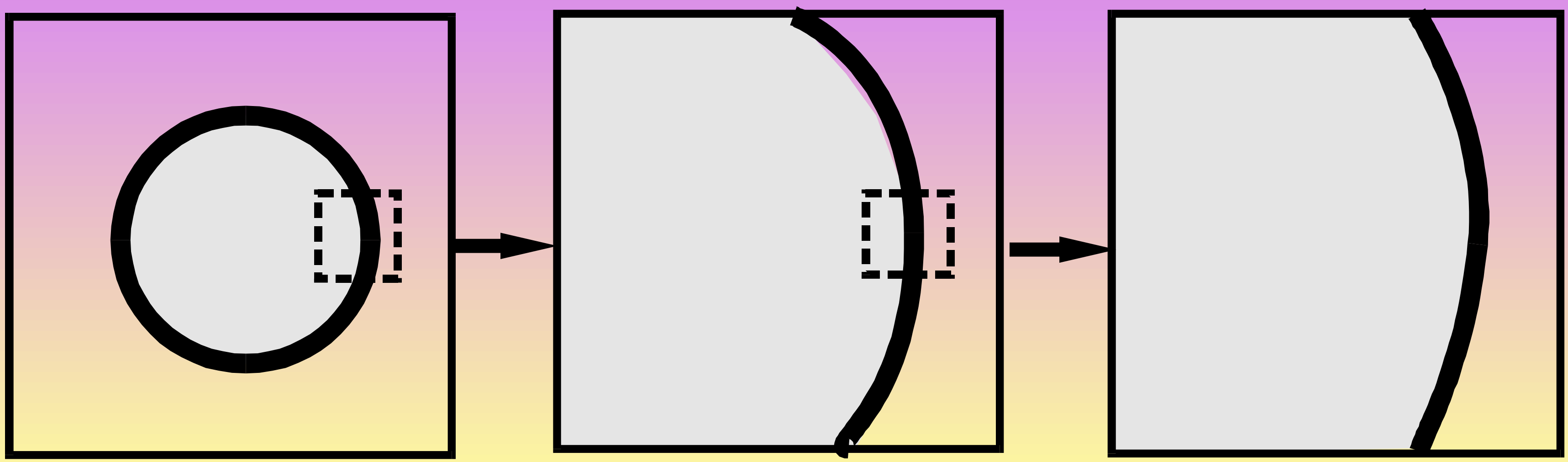
Similar, self-similar and self-affine

- It is important to distinguish between the terms similar and self-similar.

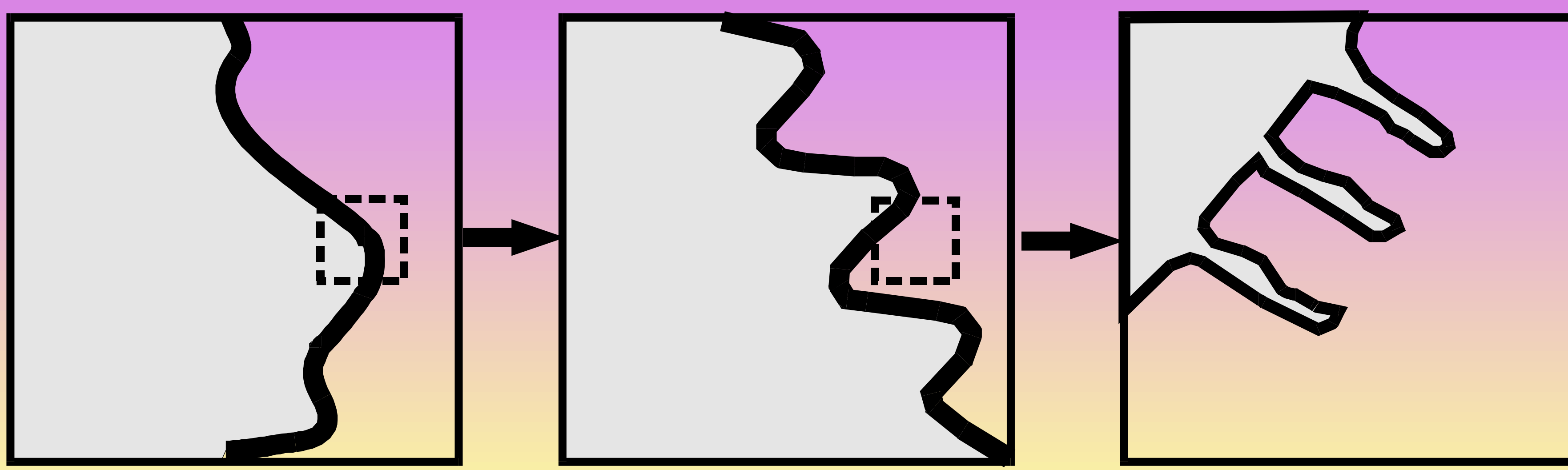
- A photograph of a face and its enlargement have the same shape and are called similar.

- However, a small portion of the photograph, e.g., the glasses, when magnified does not look like the original face in the photograph.

Non-Fractal



Fractal

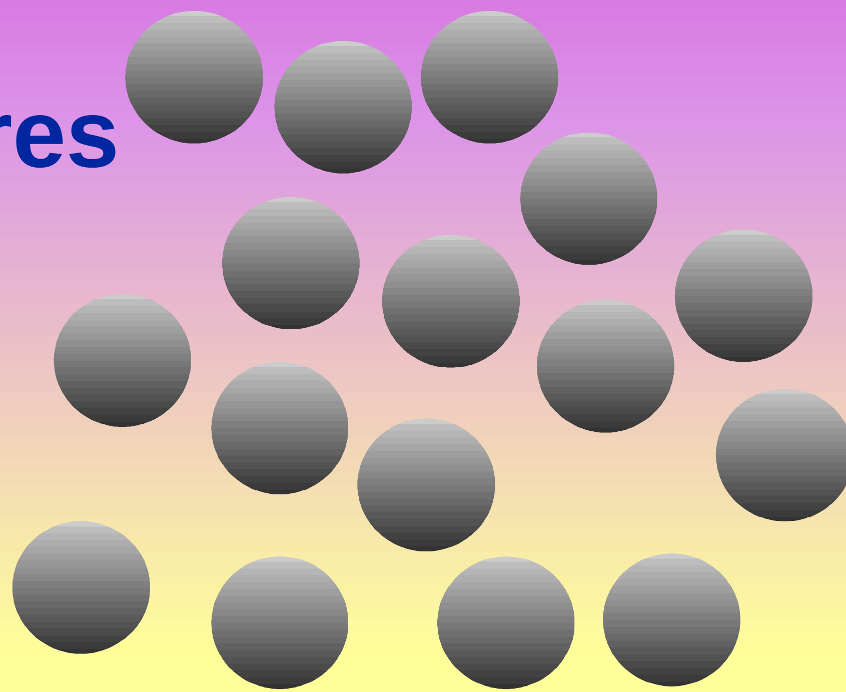


Non - Fractal

Size of Features

1 cm —

1 characteristic scale



Fractal

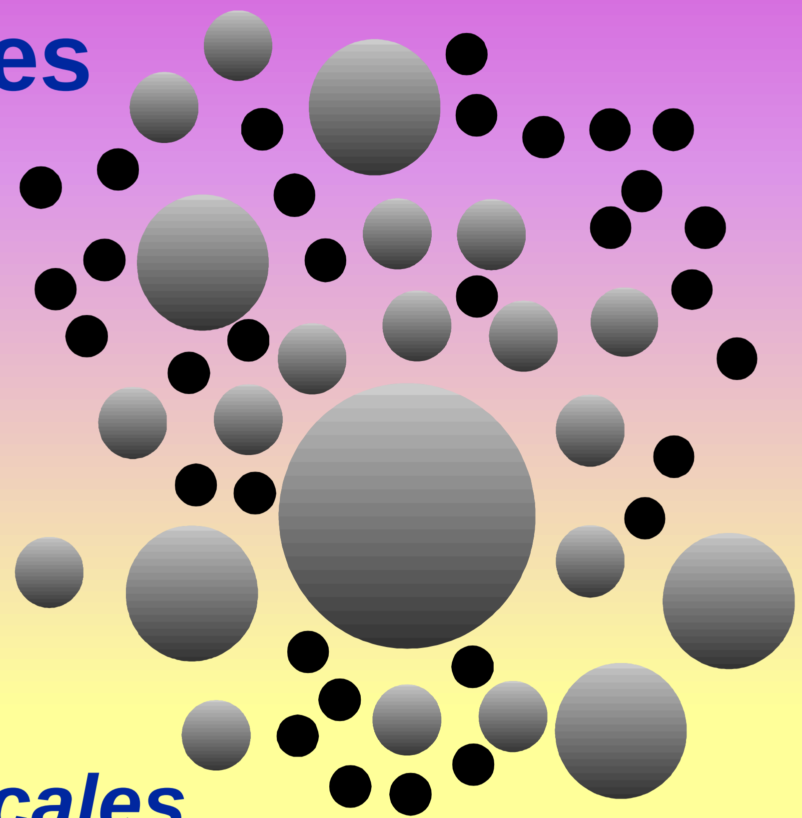
Size of Features

2 cm

1 cm

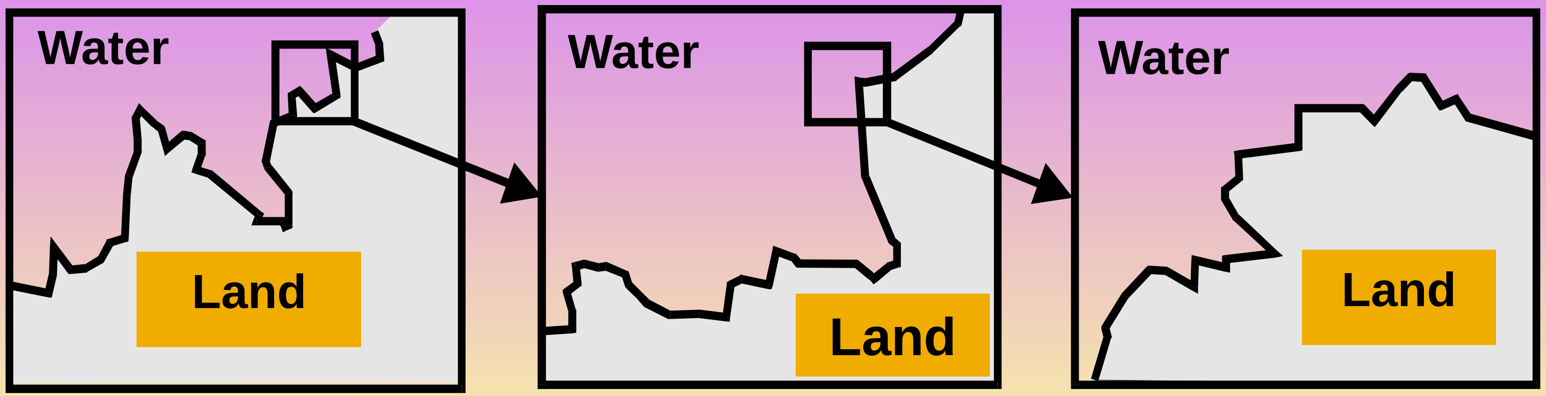
1/2 cm

1/4 cm



many different scales

Self-Similarity



To identify Fractal Geometry, we must know the difference between

Euclidean Geometry

Topological Geometry

Fractal Geometry

Fractals

Scaling | Symmetry | Self-Similar

Properties of Fractals

- Complex geometric shapes
- Defined by simple algorithms
- Non integer dimension
- Self similarity
- Invariant under scaling

Examples describing differences between topological and Euclidean dimension

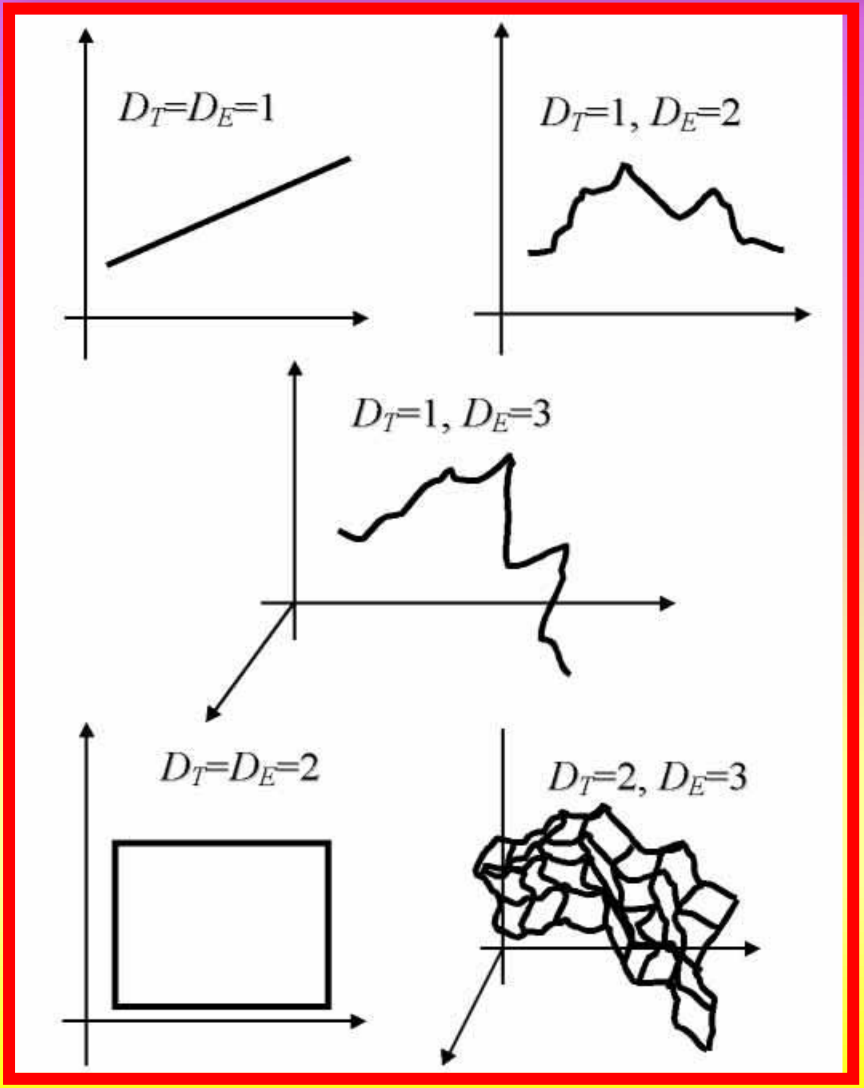
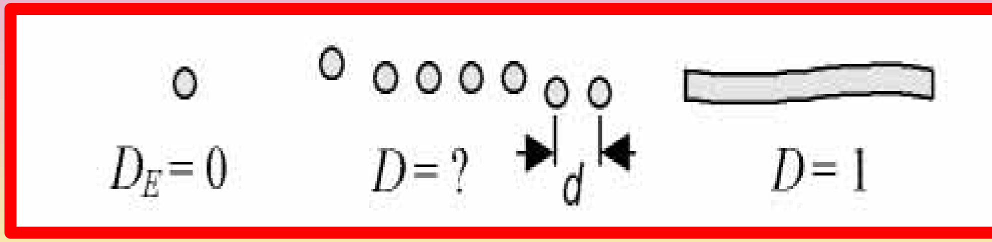
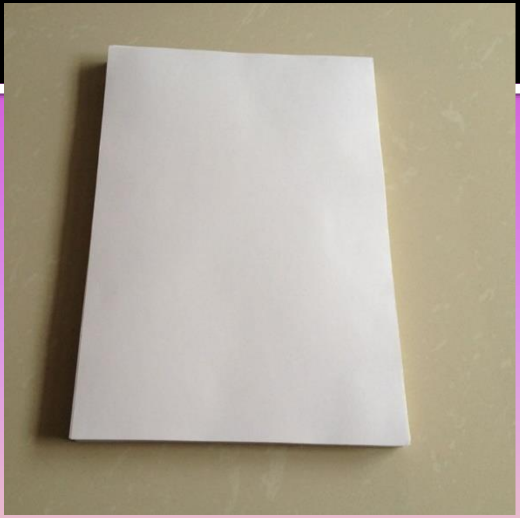
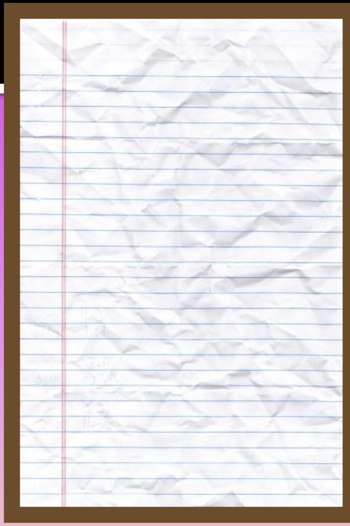


Illustration of the concept of fractal dimension







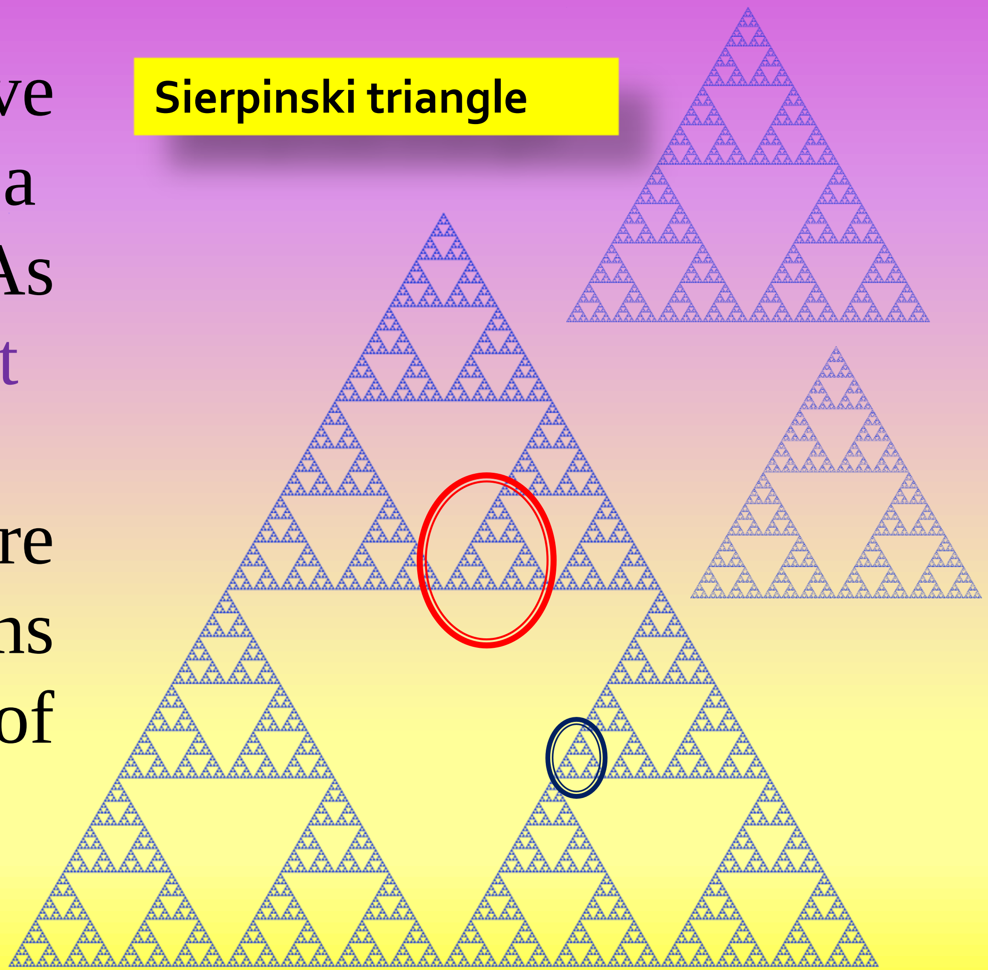
a flat paper has the Euclidean dimension $D_E=2$. If we scrunch up it into a ball-like shape, the obtained structure is a volume, having the dimension $D_E=3$. But, when we unfold a paper, the obtained structure is neither a sheet nor a ball. This structure occupies the space but not completely

self-similar

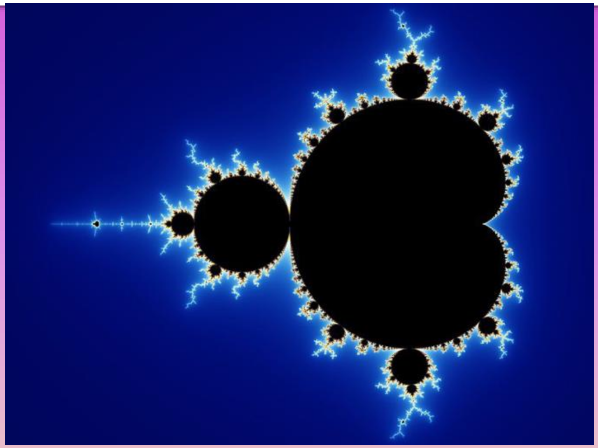
- **Self-similar** objects have similar shapes over a range of scales. As explained by Mandelbrot

- self-similar objects are formed from subregions that resemble the shape of the whole object.

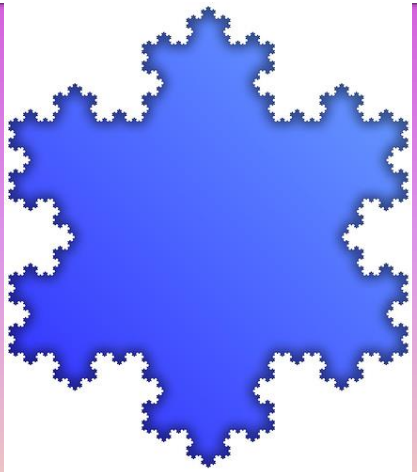
Sierpinski triangle



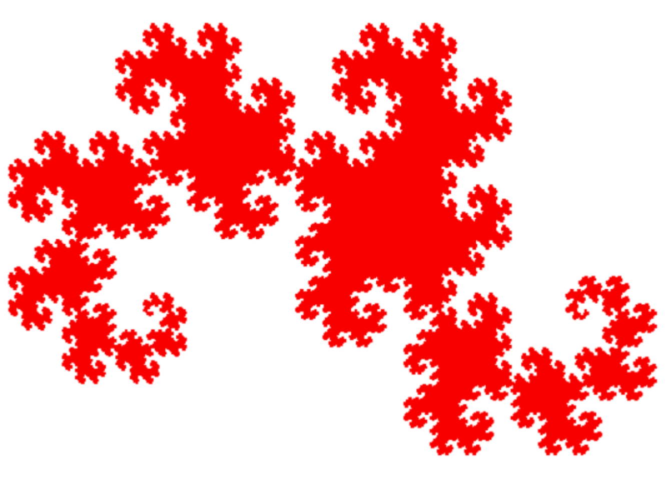
Examples of mathematical (deterministic) fractals



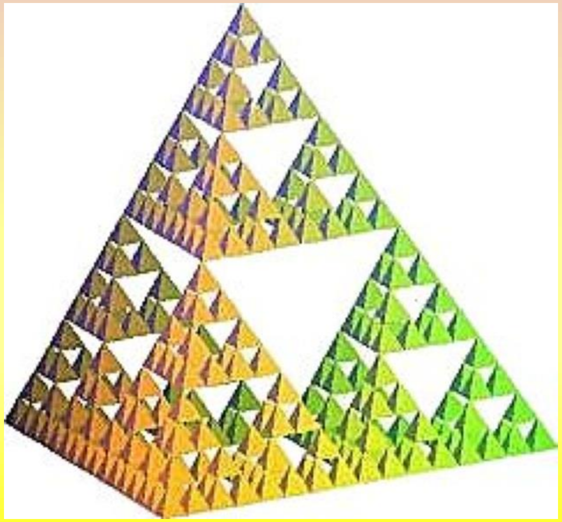
Mandelbrot set



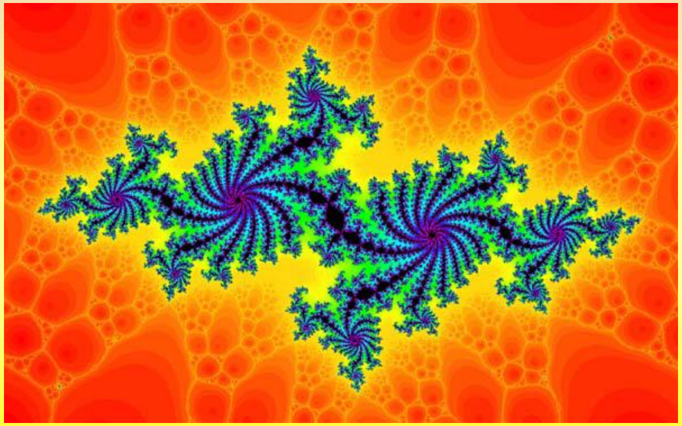
Koch snowflake



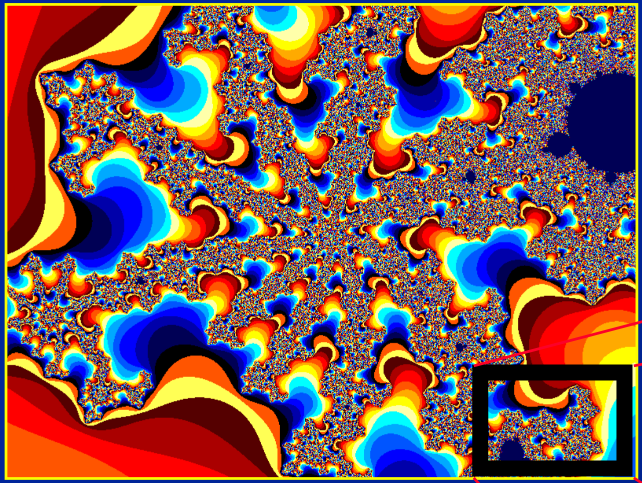
Julia set

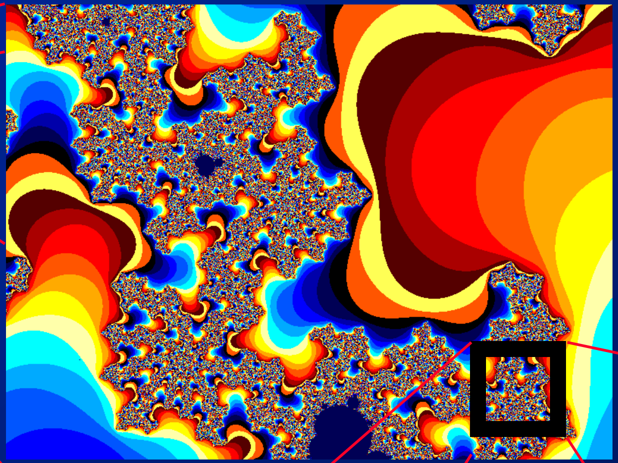


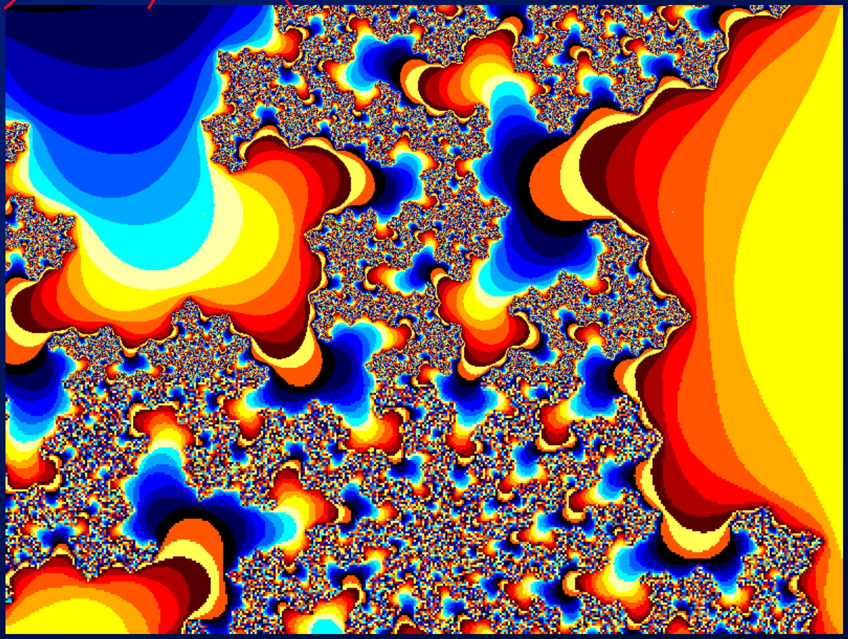
Sierpinski triangle



Julia set







...And we can continue to zoom in.   As we magnify the object, we see the same thing over and over again.....This is

Self Similarity

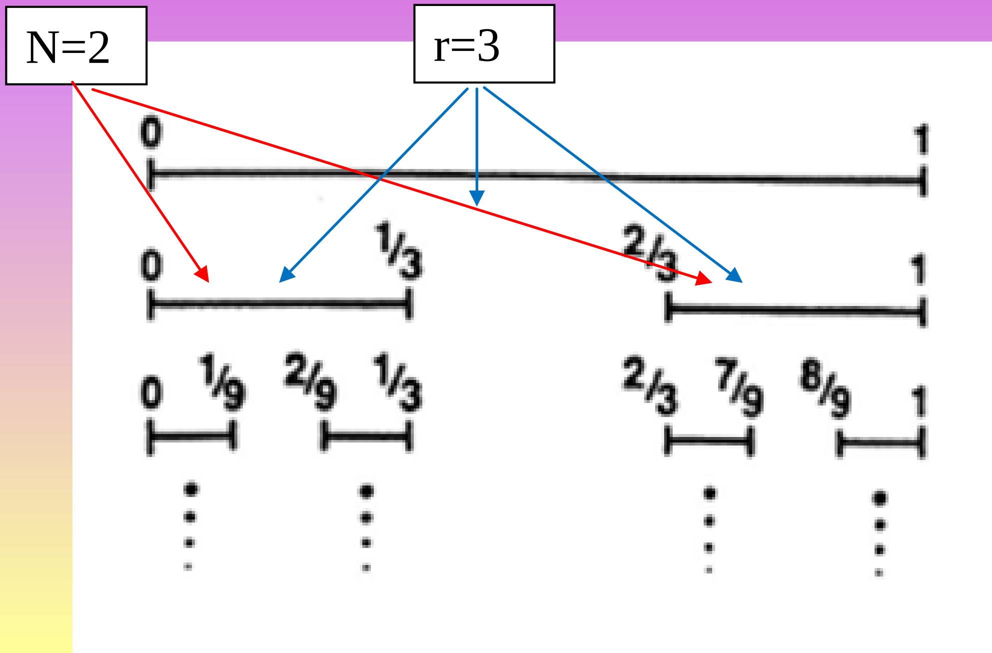
Characterization of fractal geometry

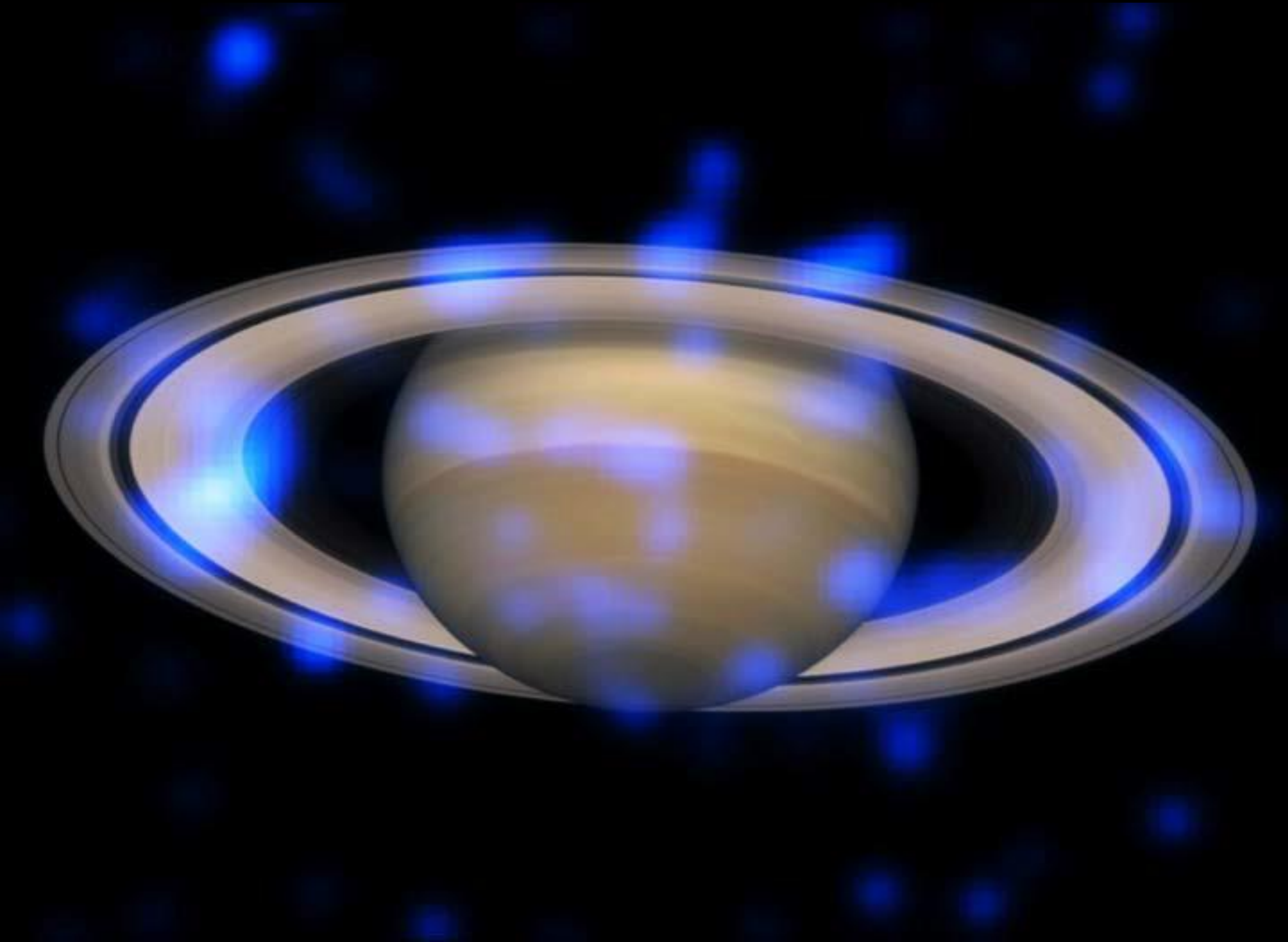
How to measure deterministic fractals?

★ Cantor set

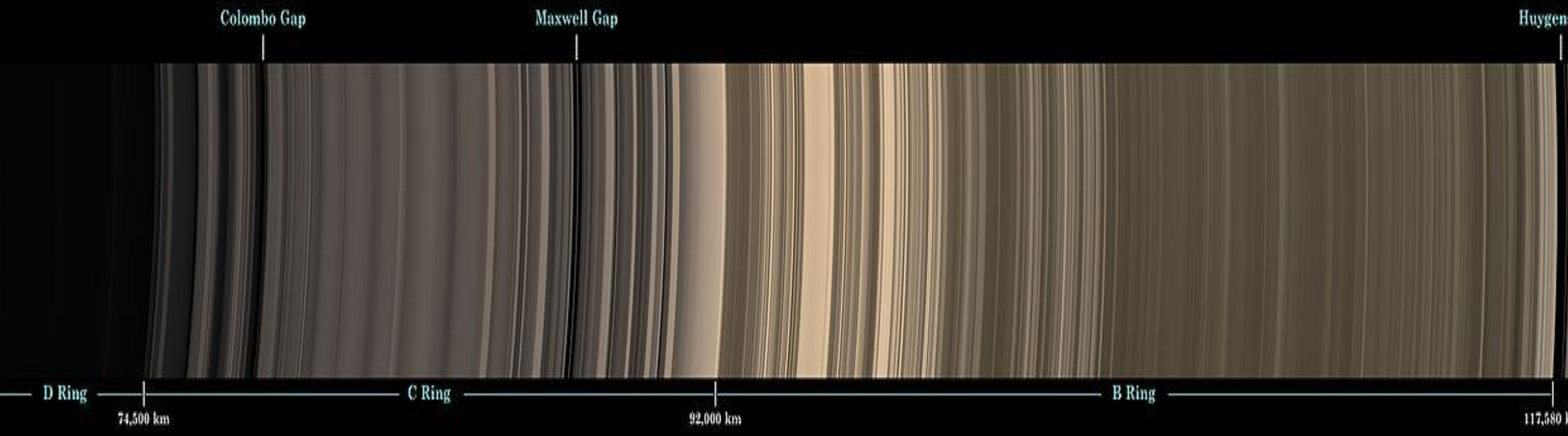
$$D = \frac{\log(N)}{\log(r)}$$

$$D = \frac{\log(N)}{\log(r)} = \frac{\log 2}{\log 3} = 0.6309$$



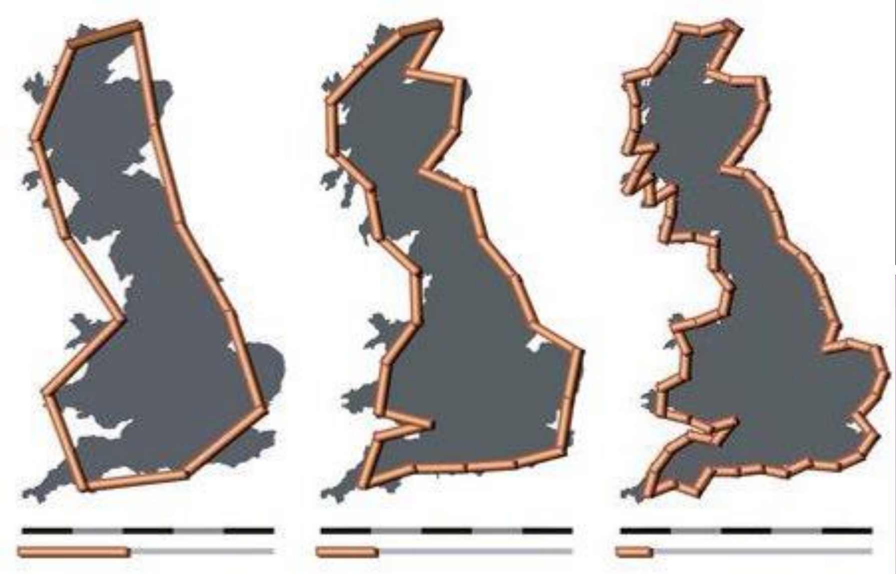


Saturn's Rings



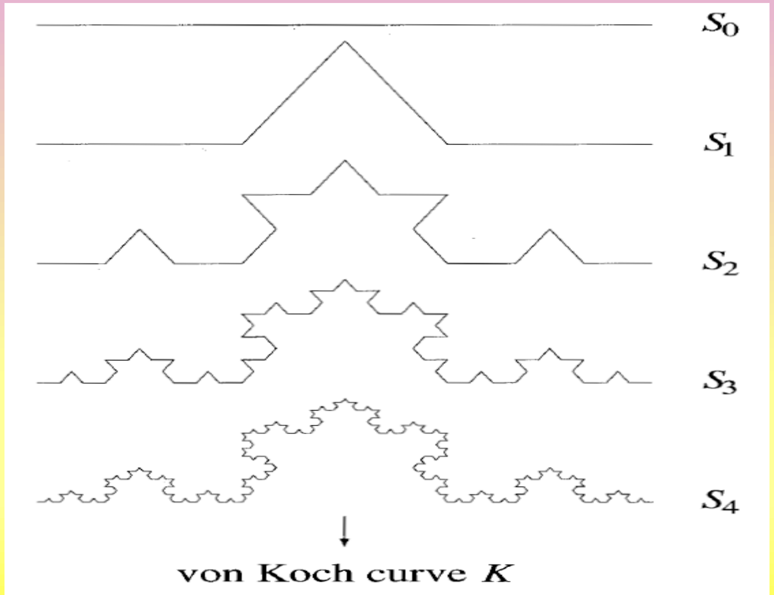
- Here is a full map (made from images) of the A-D and F rings of Saturn, with a number of named gaps shown.
- There are debates about the origin of the "main" rings A-D. These rings are very bright, and made of nearly pure water ice.
- At right is the E ring, with Enceladus shown as the source of the ring. This is also water ice, but of microscopic particles.

Koch curve



- The **Koch curve** is obtained as follows:
- start with a line segment S0.
- To generate S1, delete the middle 1/3 part of S0 and replace it with two other 2-sides of an equilateral triangle.
- Subsequent stages are generated recursively by the same rule.
- The limit K=S∞

is **the von Koch curve**.



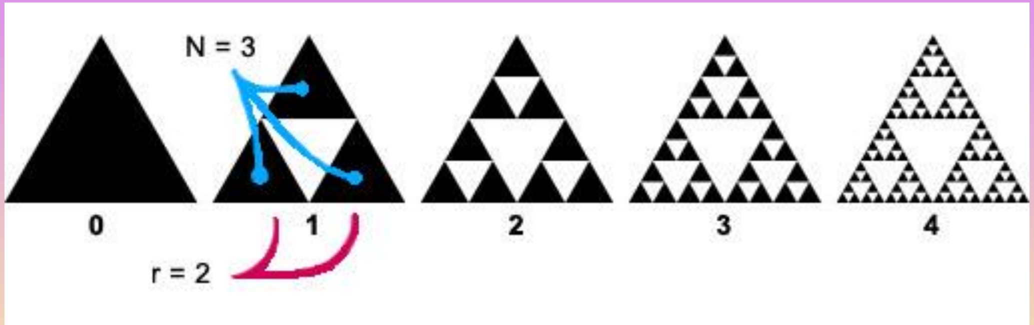
$$D = \frac{\log(N)}{\log(r)} = \frac{\log 4}{\log 3} = 1.2619$$

Sierpinski triangle

1-Start with any triangle in a plane.

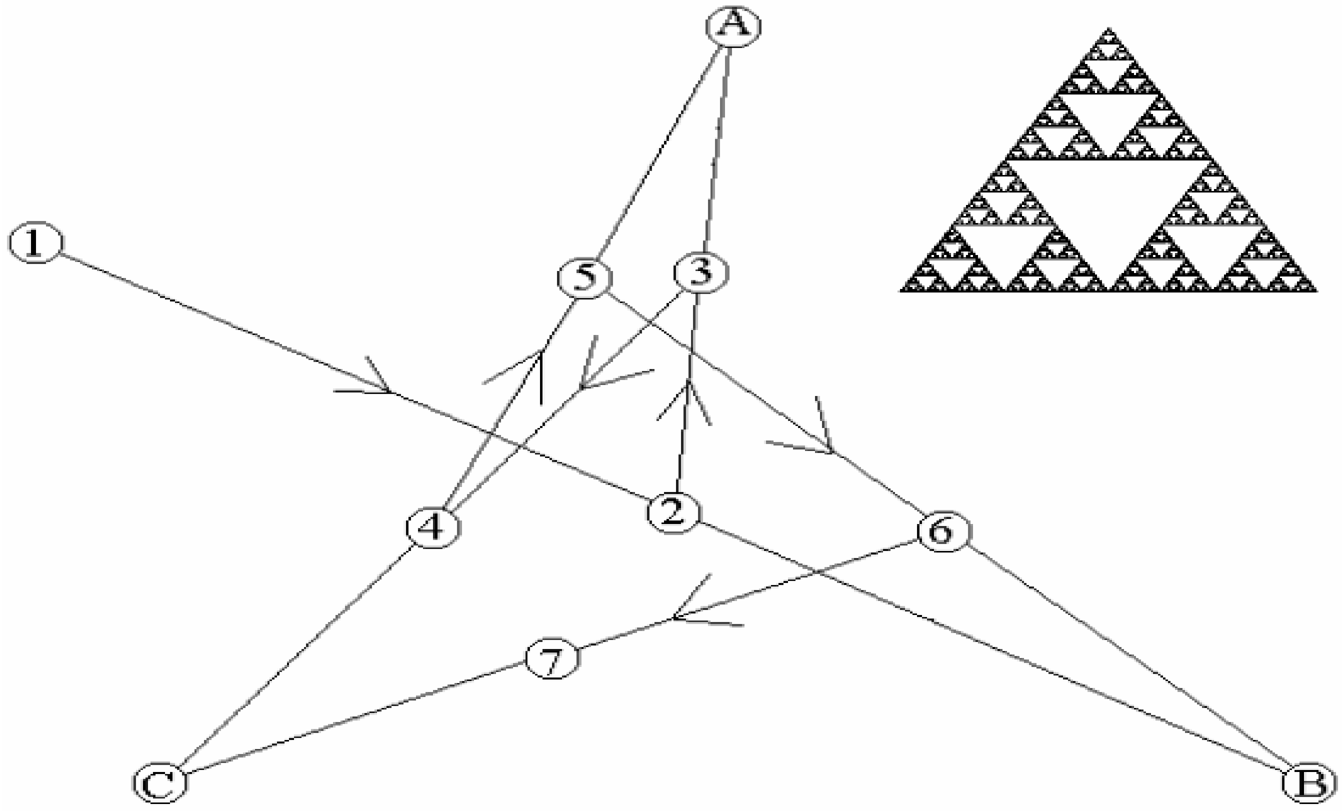
2-Shrink the triangle to 1/2 height and 1/2 width, make three copies.

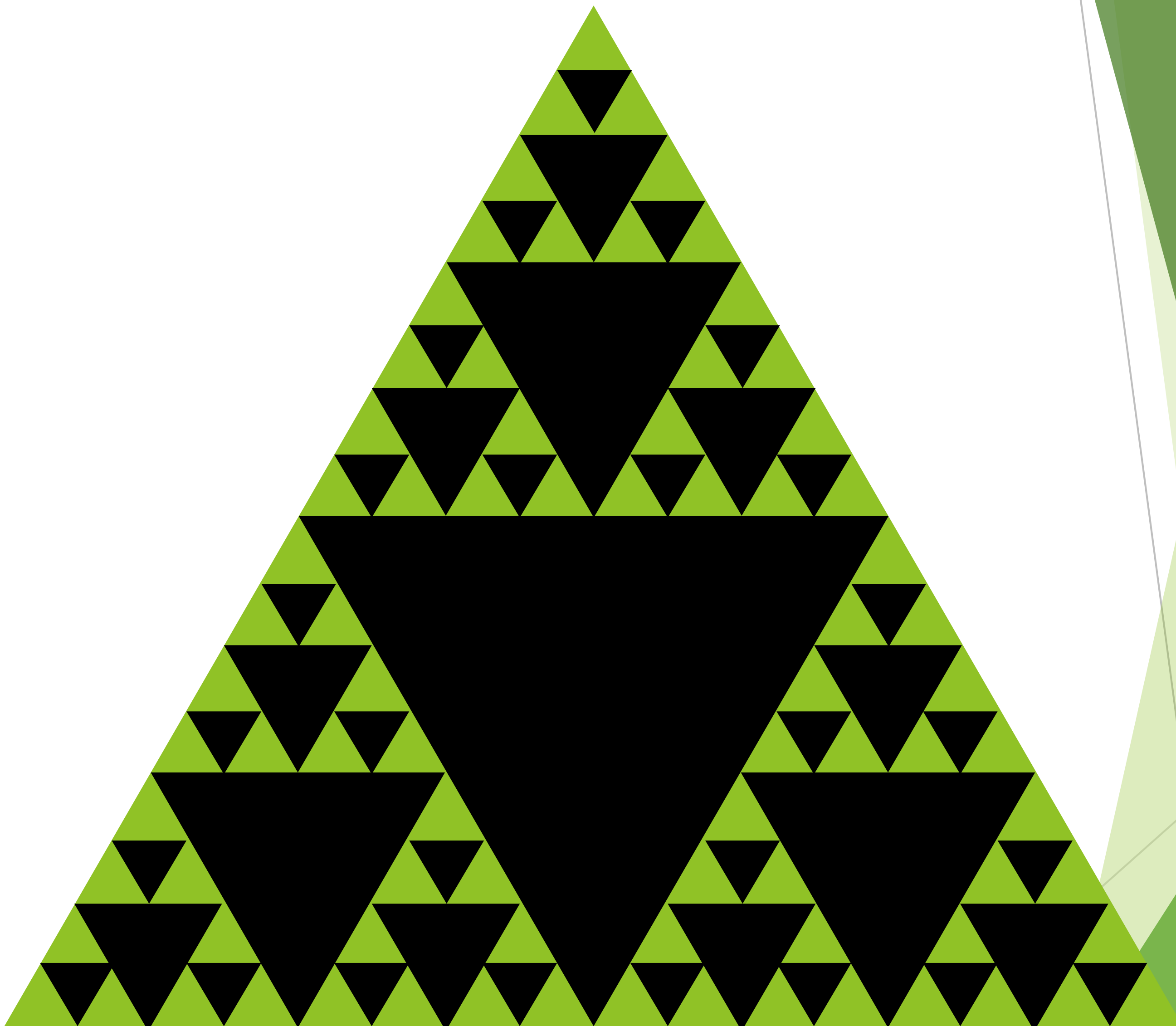
3-Repeat step 2 with each of the smaller triangles.

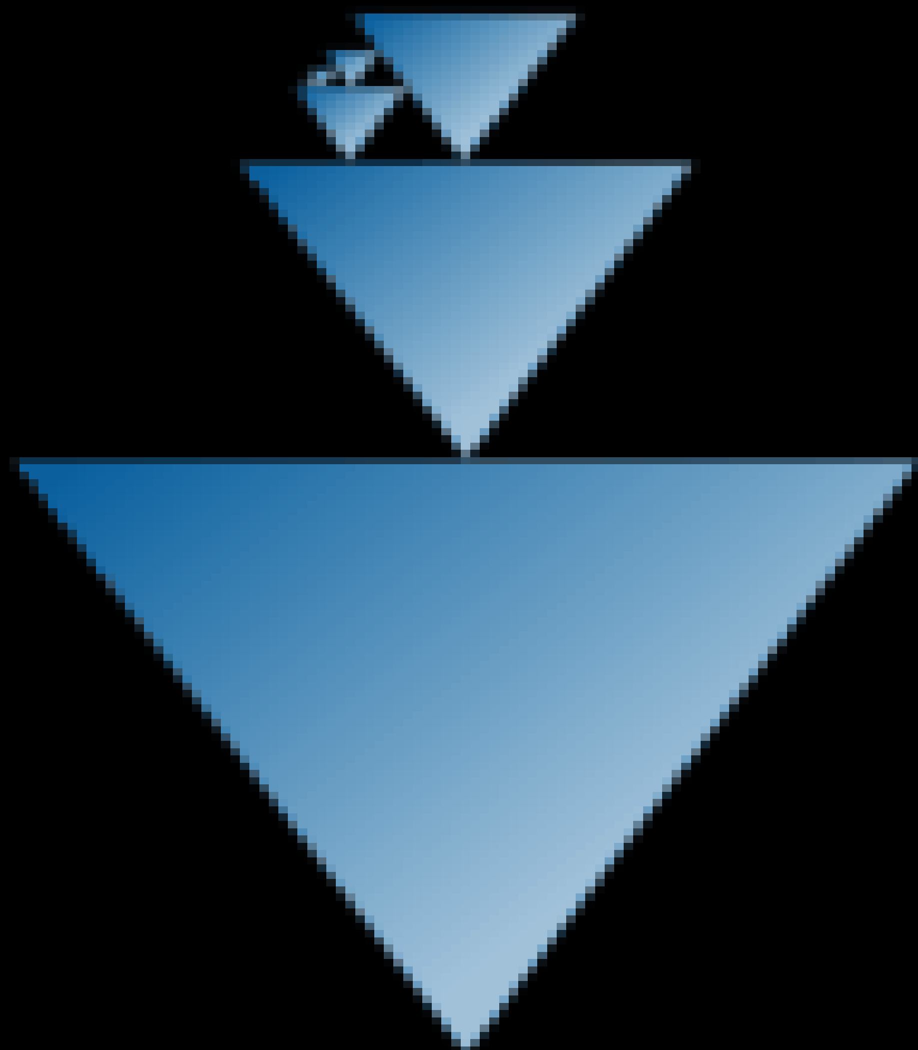


$$D = \frac{\log(N)}{\log(r)} = \frac{\log(3)}{\log(2)} = 1.585$$

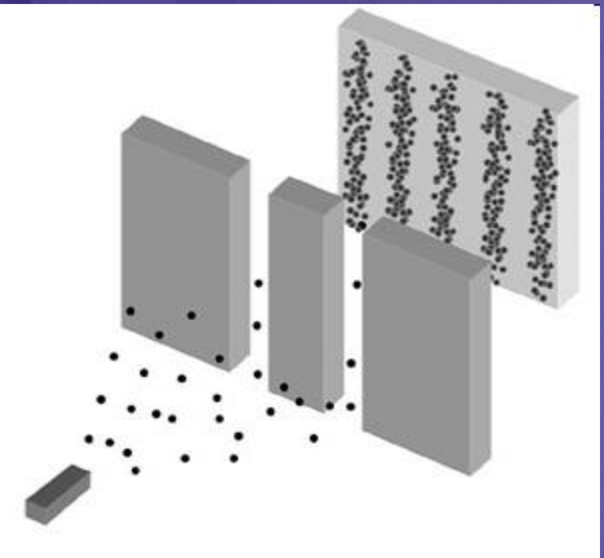
The Chaos Game

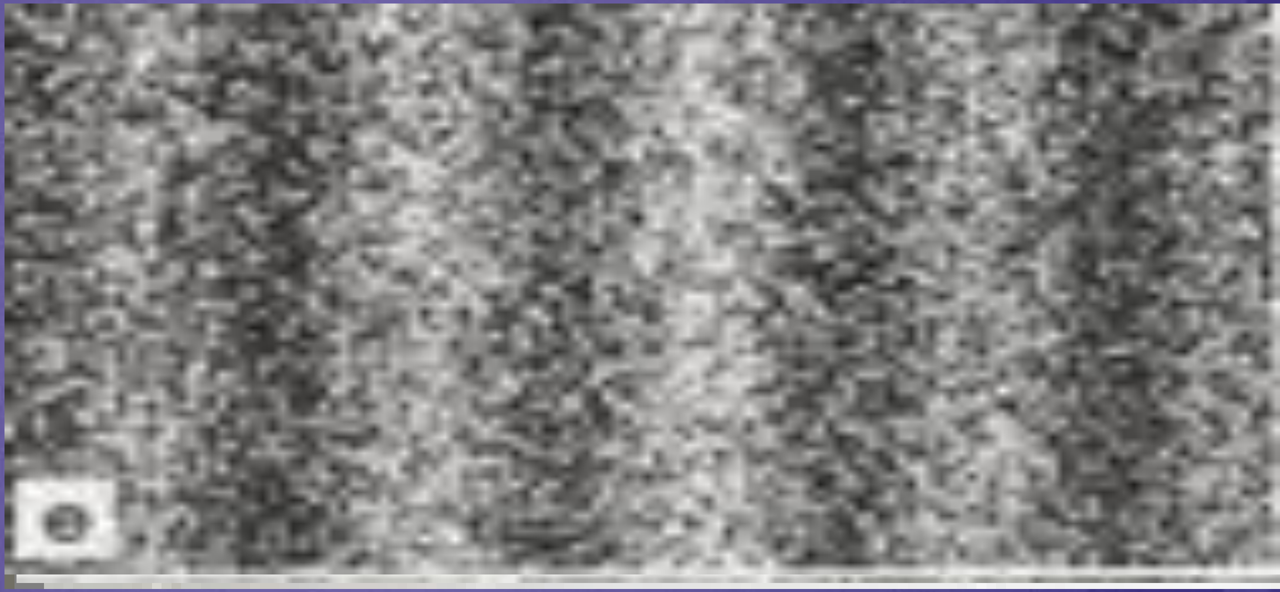






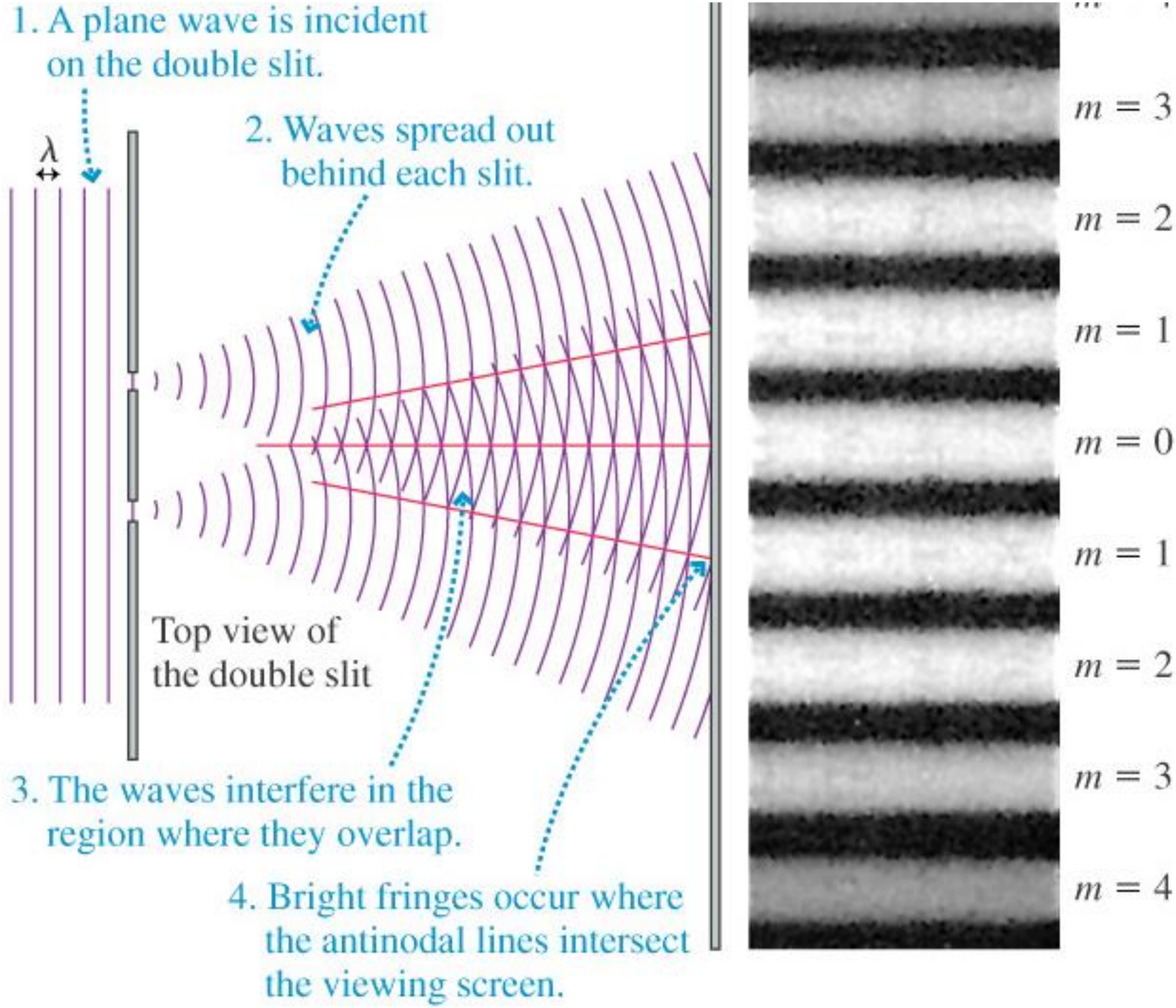
The Most Beautiful Experiment in Physics



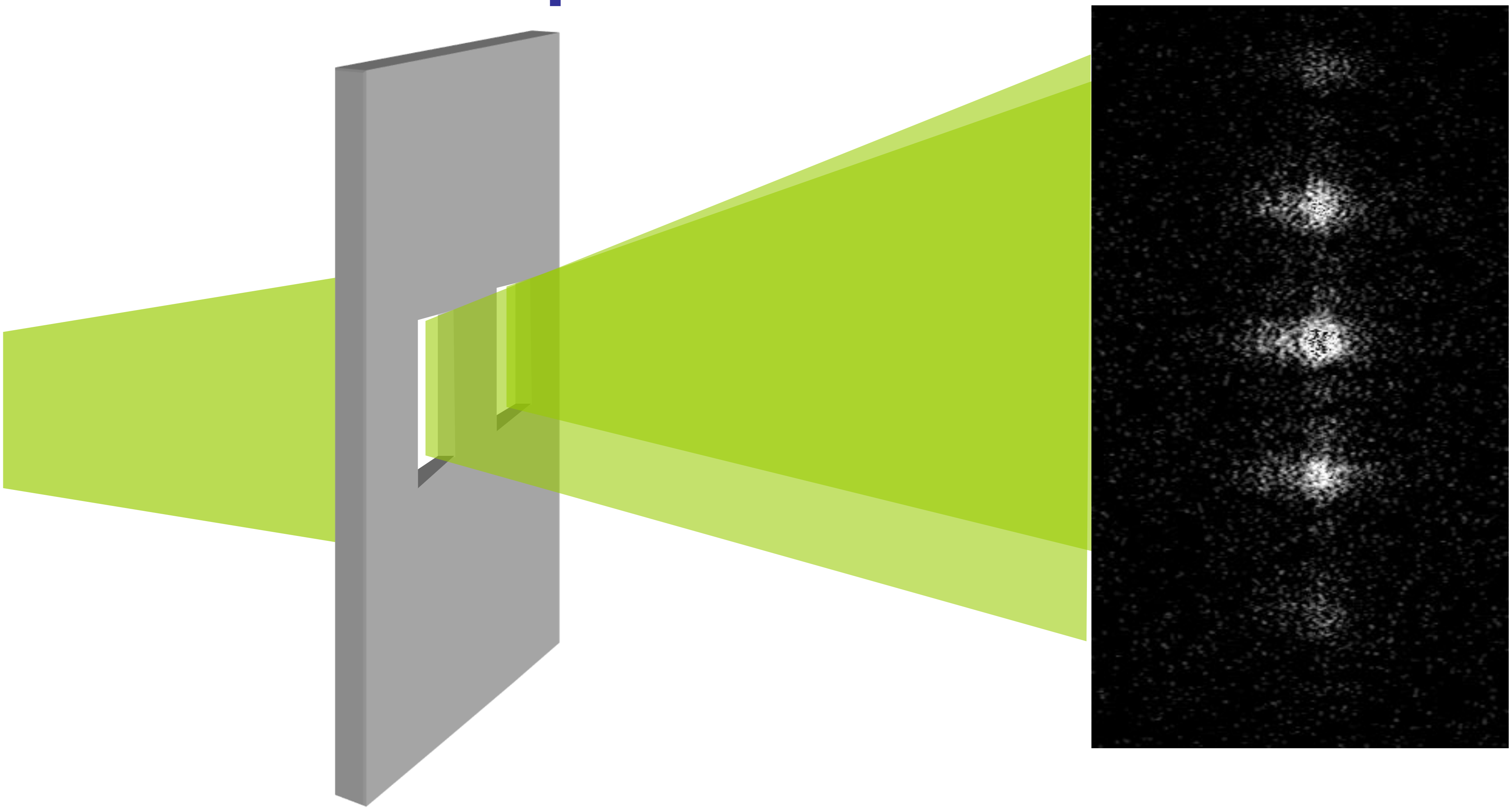


"We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery." R.Feynman

Light is a wave: Two slit interference

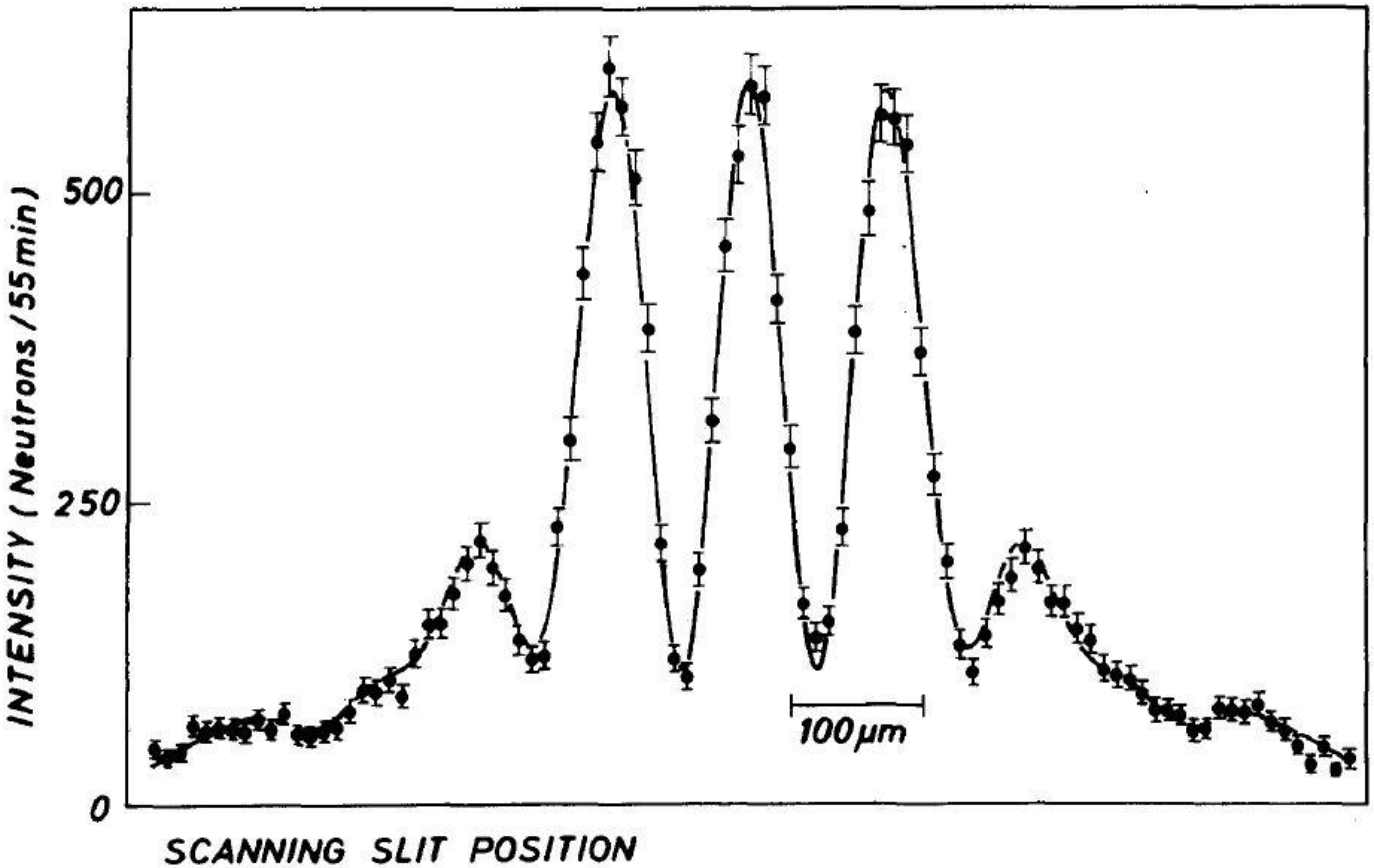


Double-slit particle interference

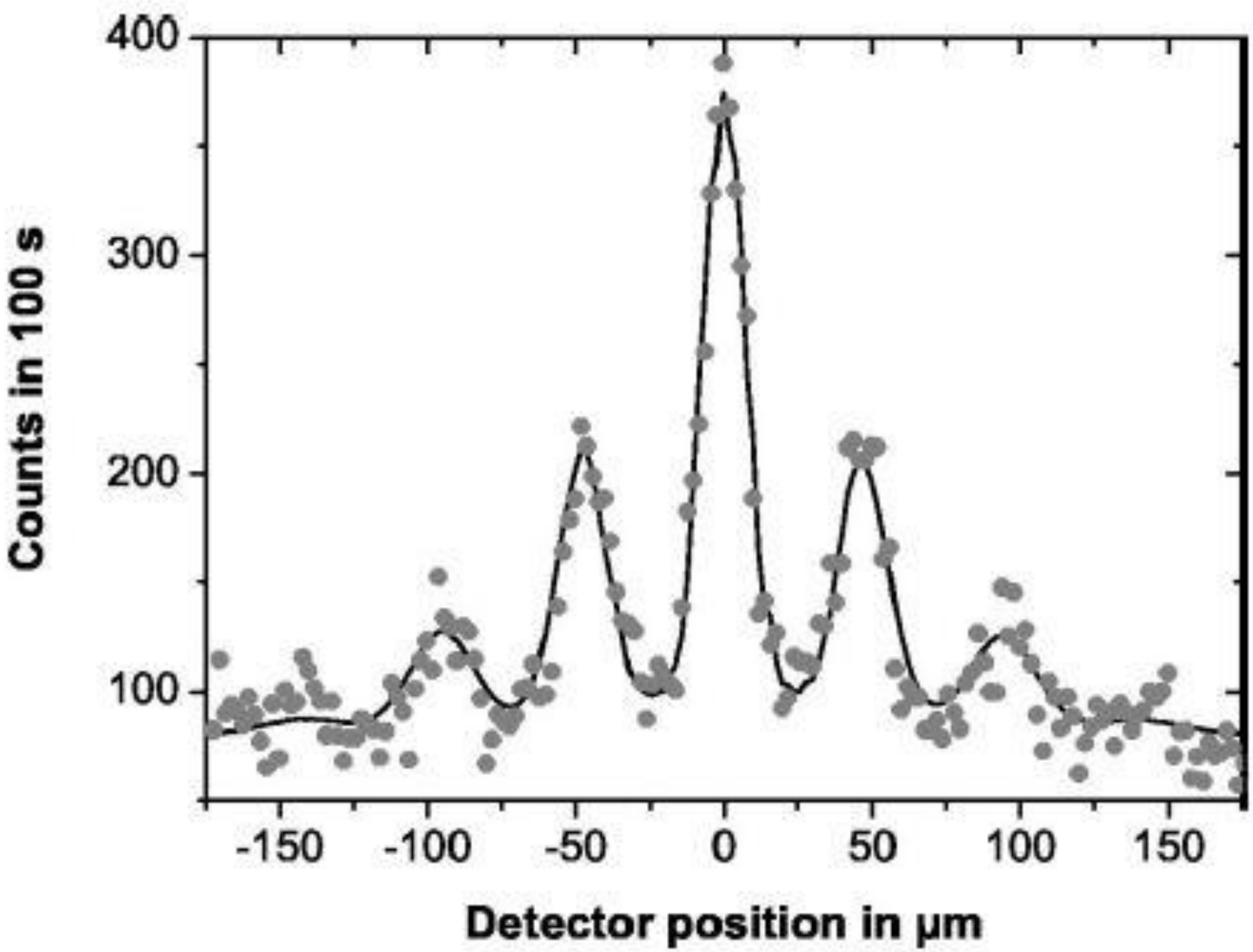


- With single photons at a time
- Which slit does the photon go through?

Neutron Interference

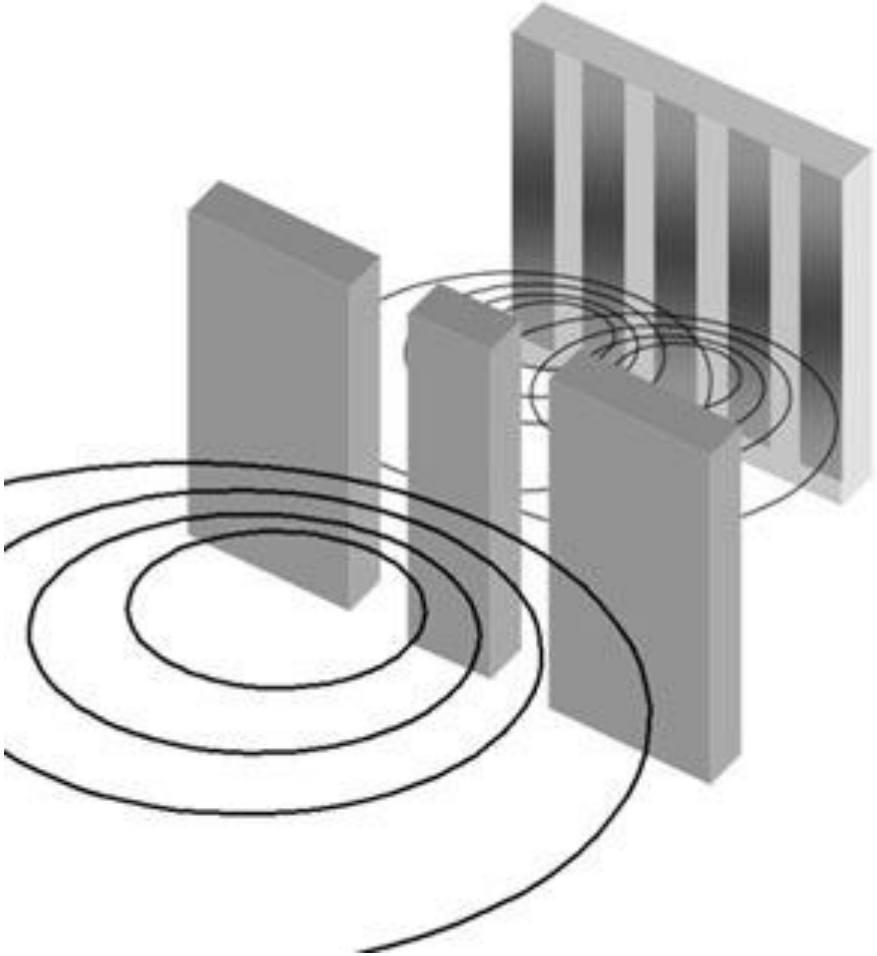


C-60 Interference



Quantum Superposition

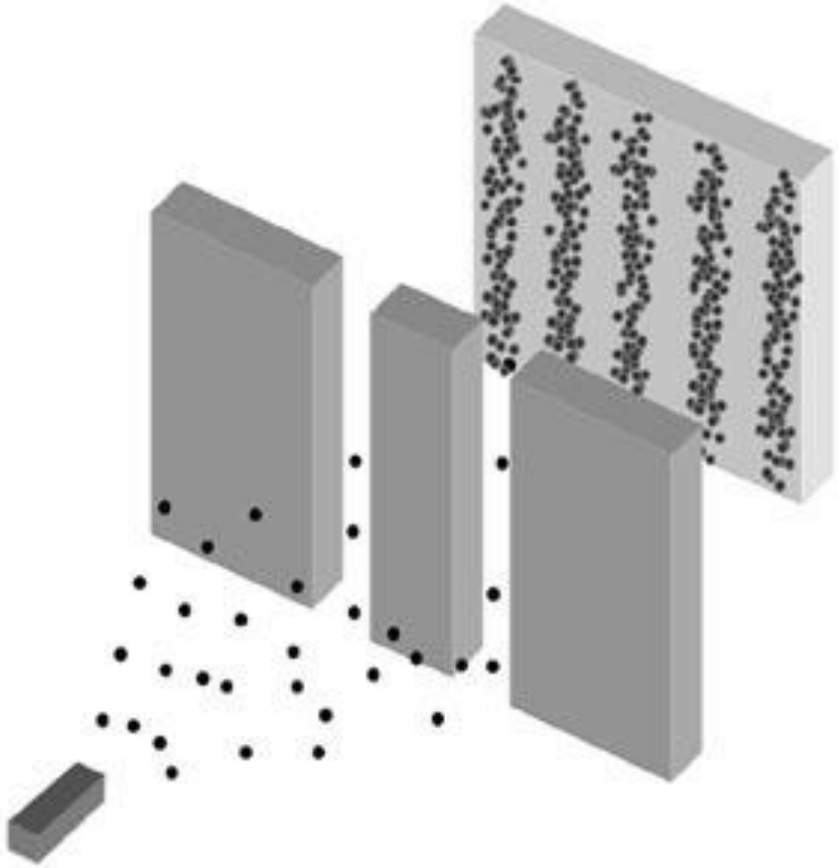
The double slit experiment



Interference of waves

Quantum Superposition

The double slit experiment

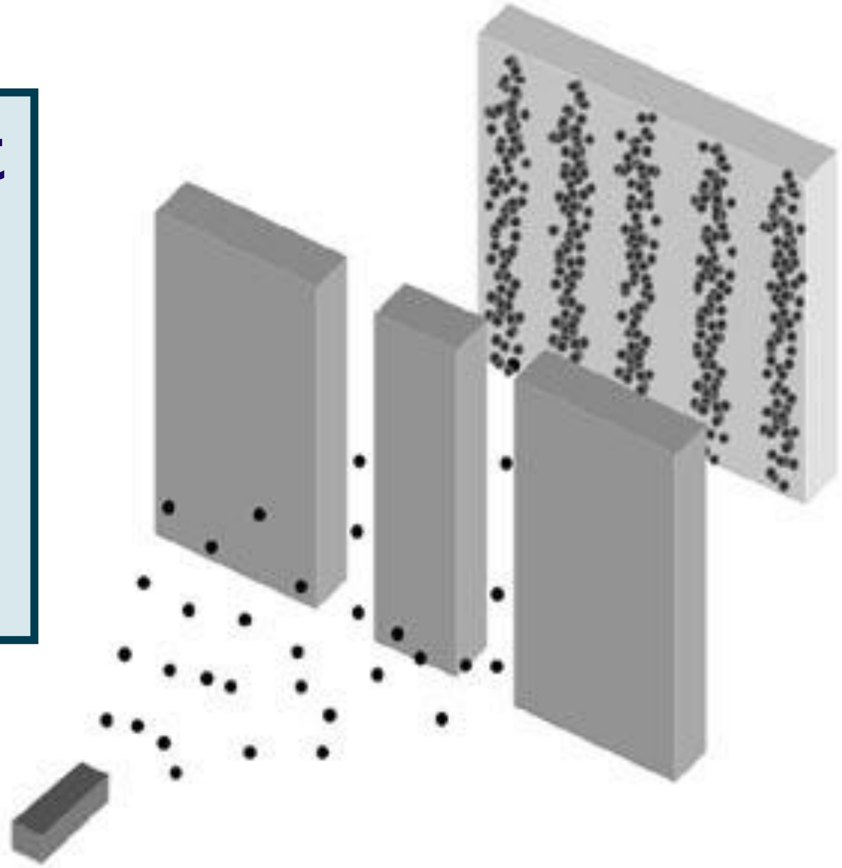


Interference of electrons

Quantum Superposition

The double slit experiment

Which slit does an electron pass through ?

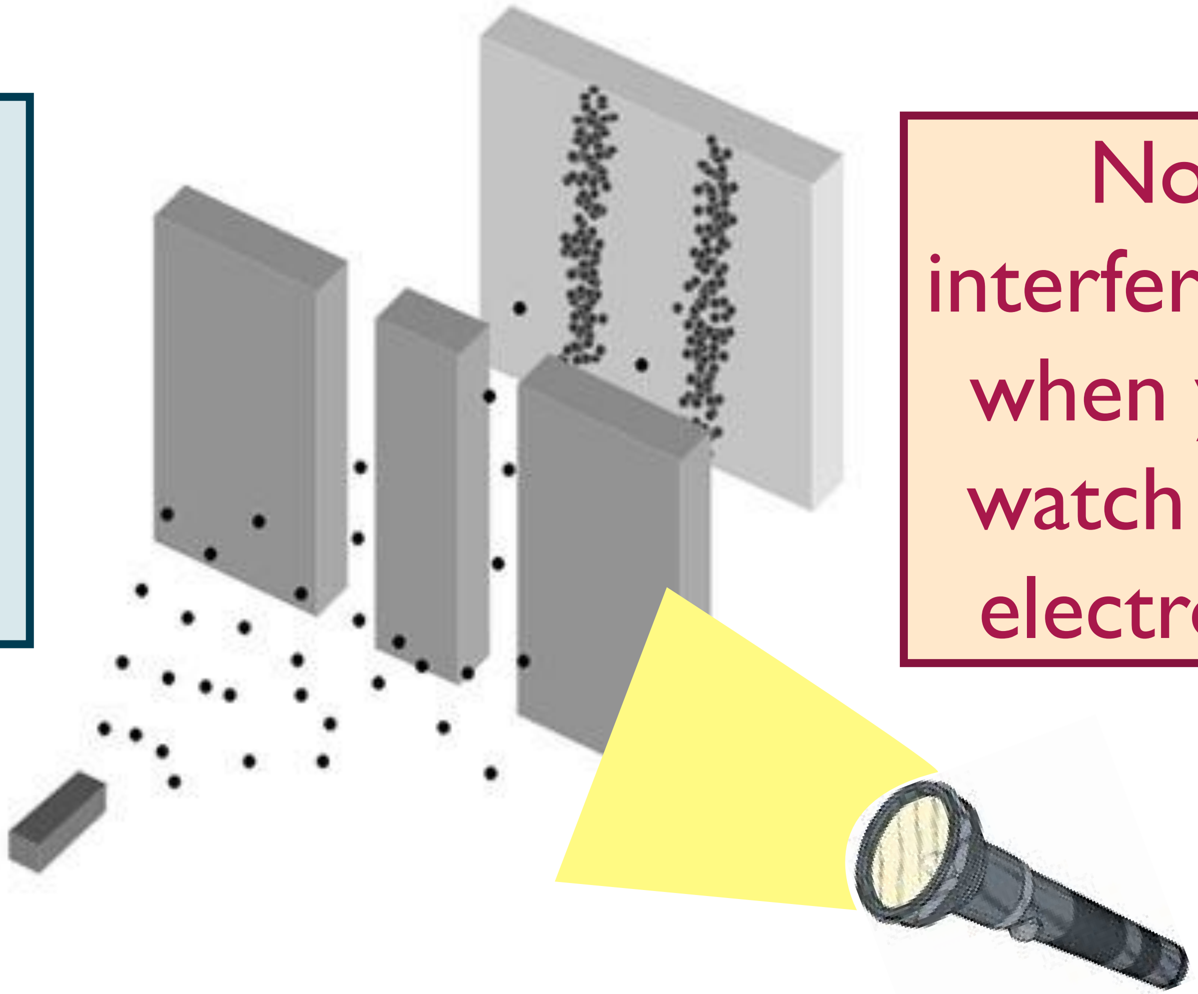


Interference of electrons

Quantum Superposition

The double slit experiment

Which slit does an electron pass through ?



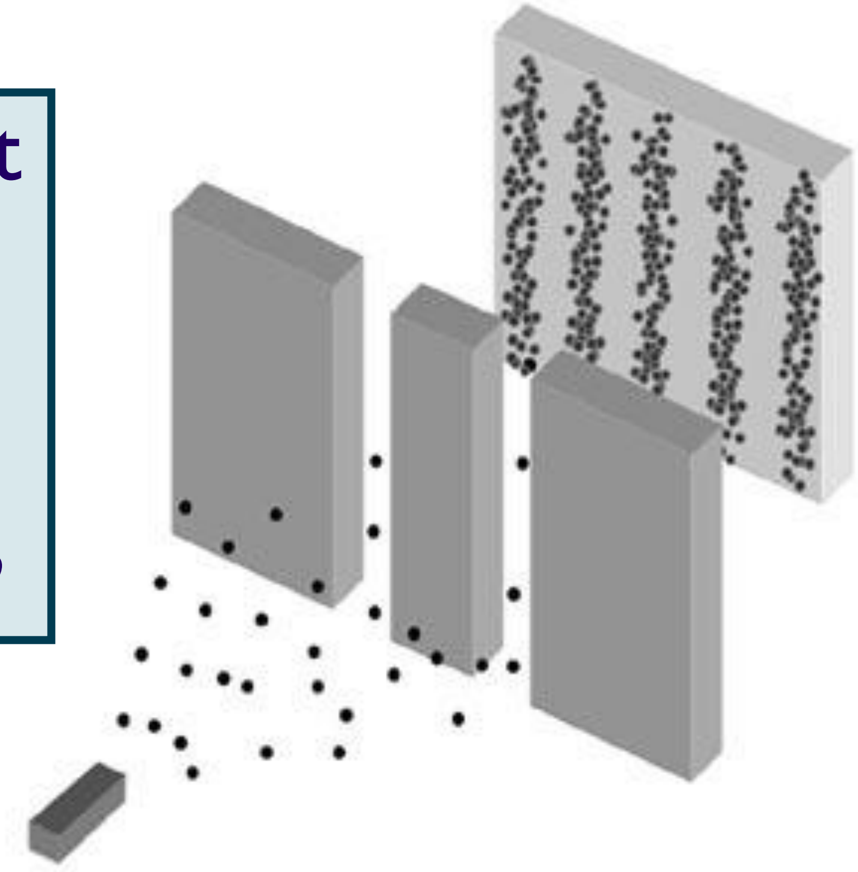
No interference when you watch the electrons

Interference of electrons

Quantum Superposition

The double slit experiment

Which slit does an electron pass through ?

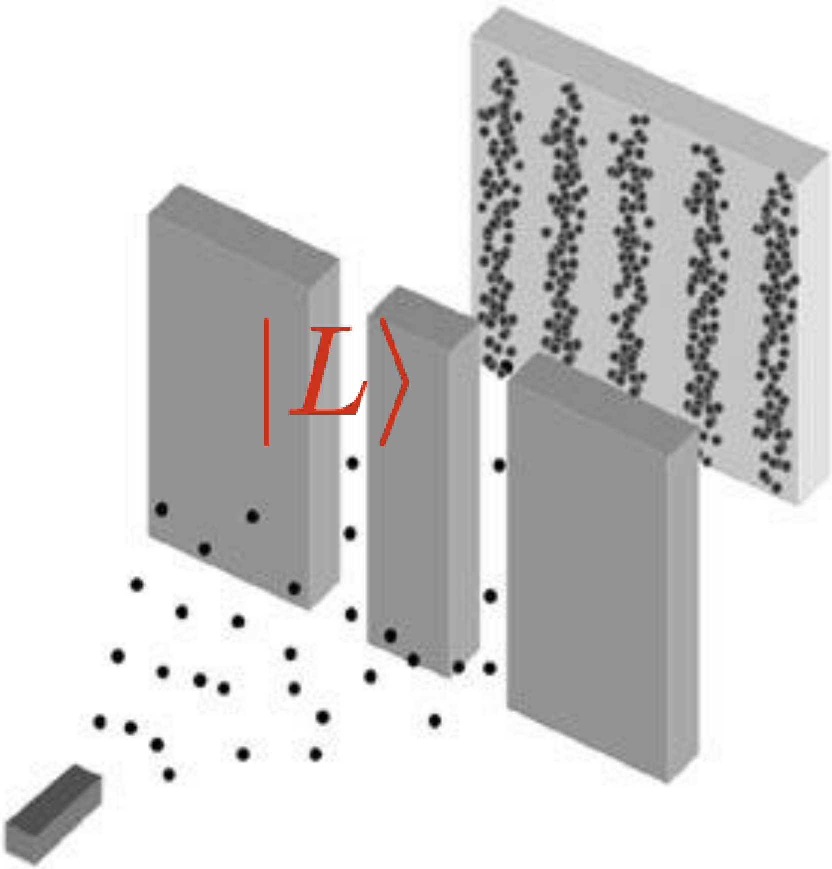


Each electron passes through both slits !

Interference of electrons

Quantum Superposition

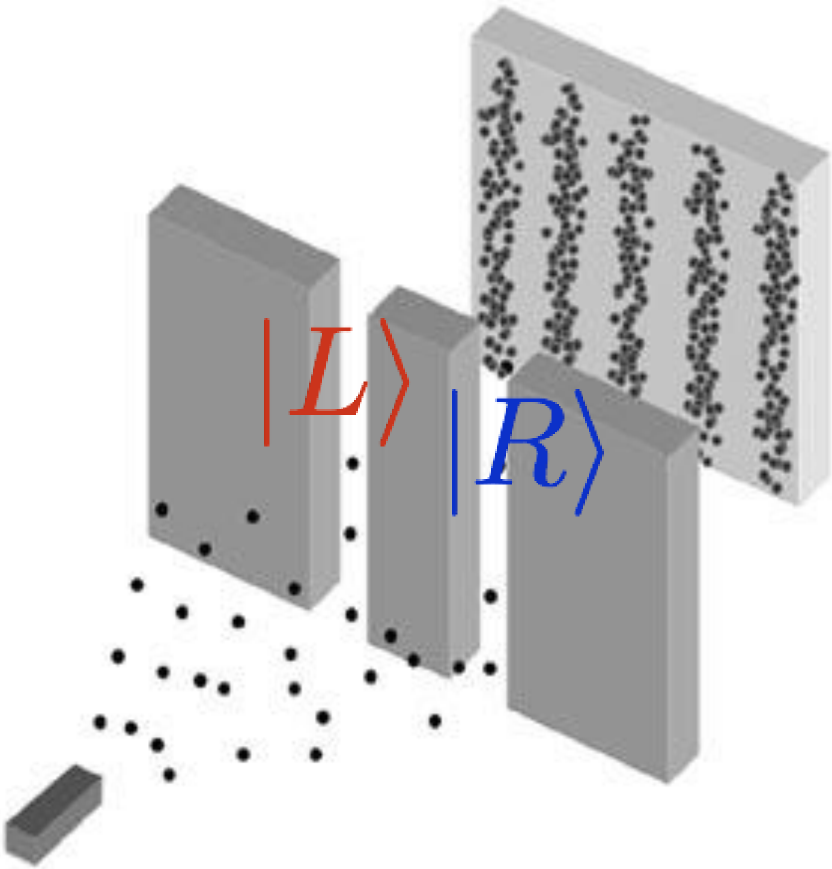
The double slit experiment



Let $|L\rangle$ represent the state with the electron in the left slit

Quantum Superposition

The double slit experiment

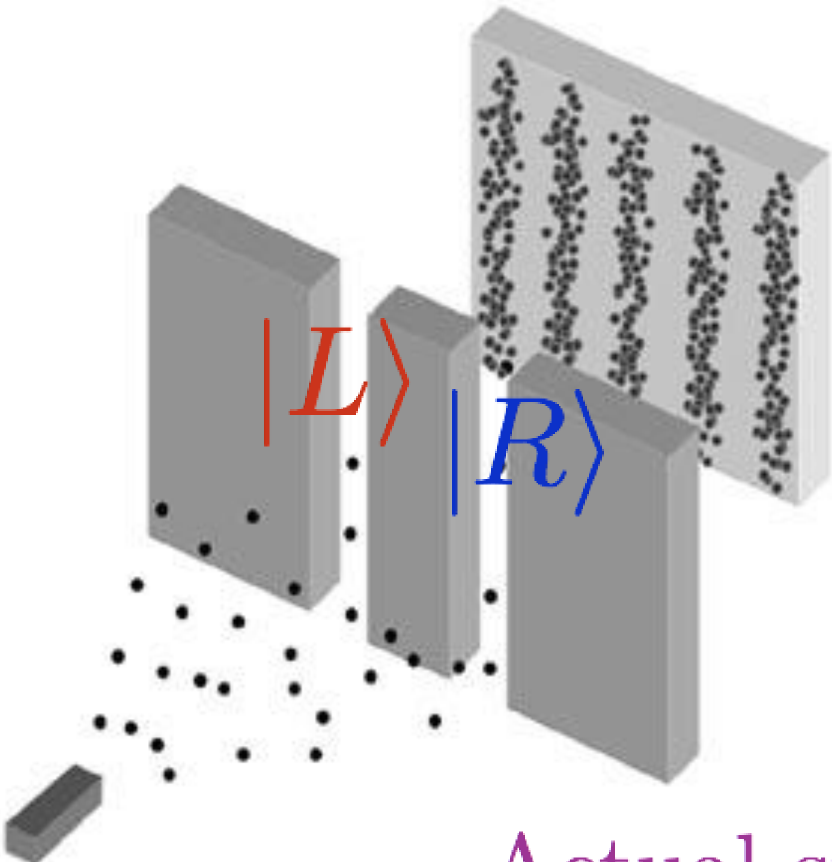


Let $|L\rangle$ represent the state with the electron in the left slit

And $|R\rangle$ represents the state with the electron in the right slit

Quantum Superposition

The double slit experiment



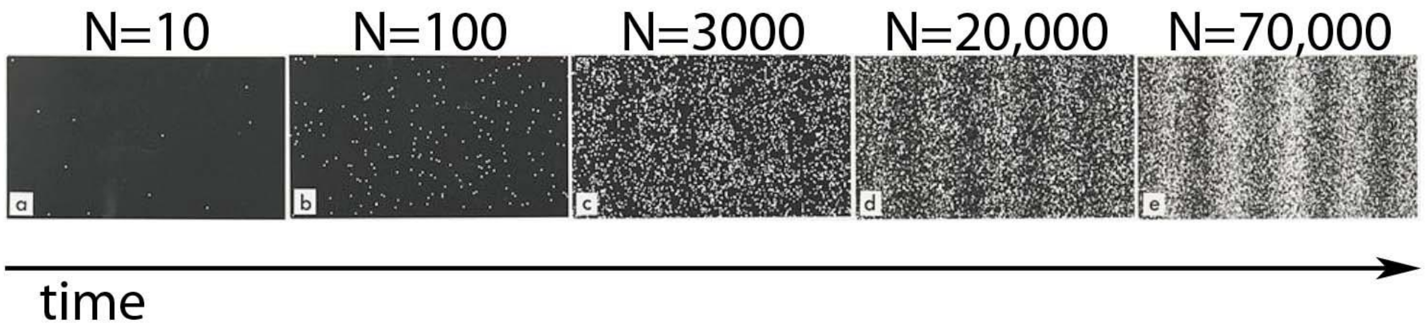
Let $|L\rangle$ represent the state with the electron in the left slit

And $|R\rangle$ represents the state with the electron in the right slit

Actual state of the electron is

$$|L\rangle + |R\rangle$$

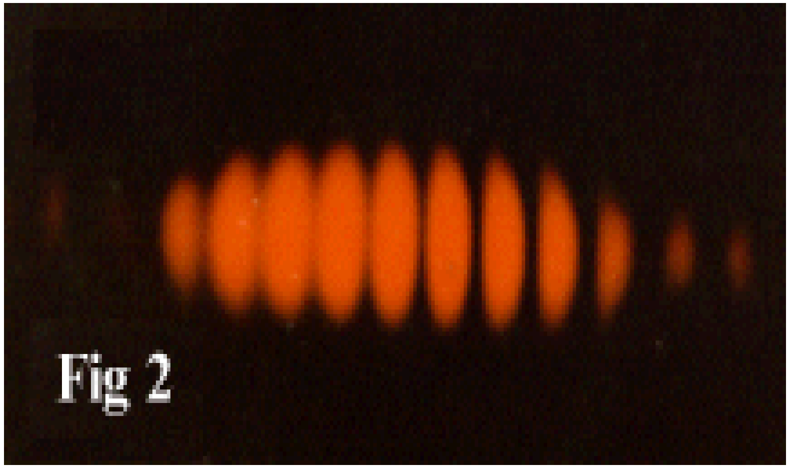
Double-Slit Experiment

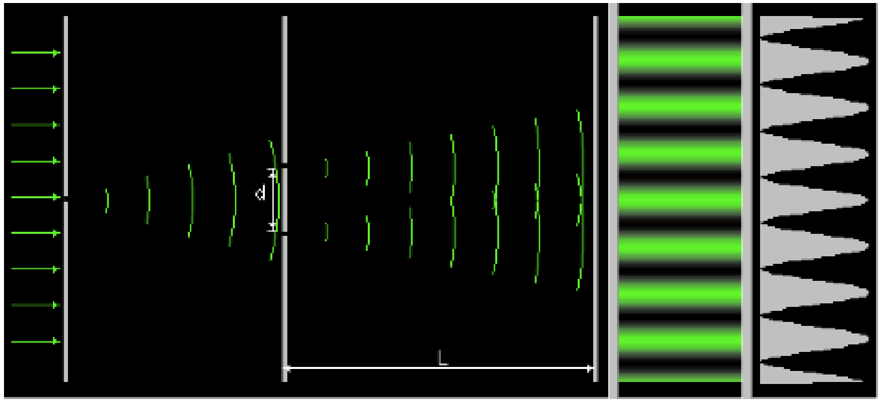


- Pass a beam of electrons through a double-slit apparatus.
- Individual electrons are detected as points on the screen.
- Over time, a fringe pattern of dark and light bands appears.

Particle-wave duality

What is this?

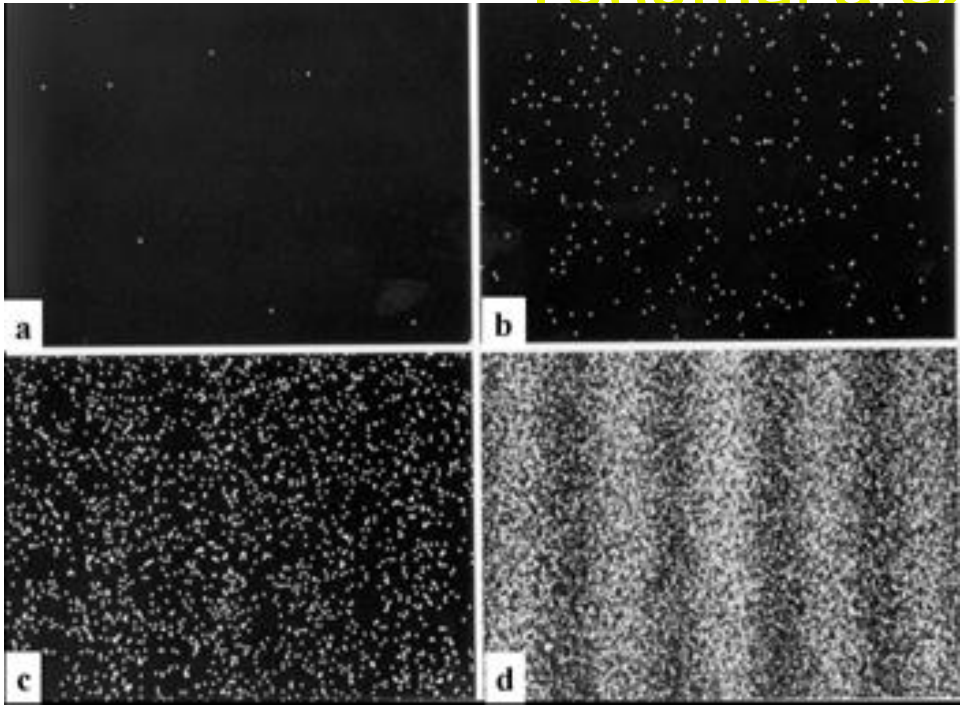




Young's "double-slit" experiment

Interference: Light consists of WAWES

What can we conclude about the electron from Tonomura experiment?

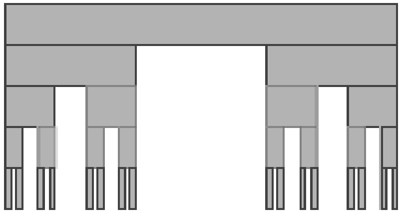


Electrons appear to have both particle-like and wave-like properties!

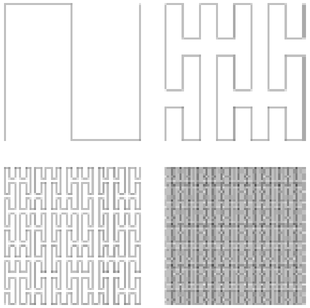
Euclidean vs. Fractal Geometry

Euclidean Geometry	Fractal Geometry
Objects defined by analytical equations	Objects defined by recursive algorithms
Localy smooth, differentiable	Localy rough, not differentiable
Elements: vertices, edges, surfaces	Elements: iteration of functions

Georg Cantor

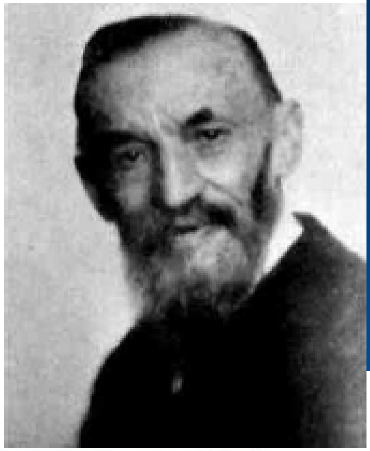


Cantor Set, 1870

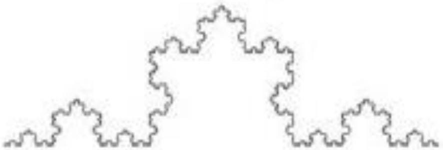


Peano Curve, 1890

Guiseppe Peano

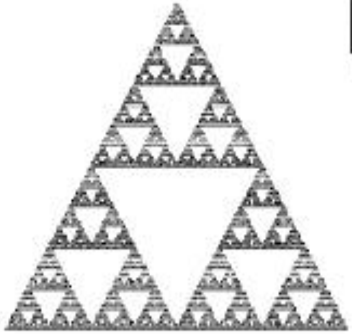


Koch Curve, 1904



Helge von Koch

Waclaw Sierpinski



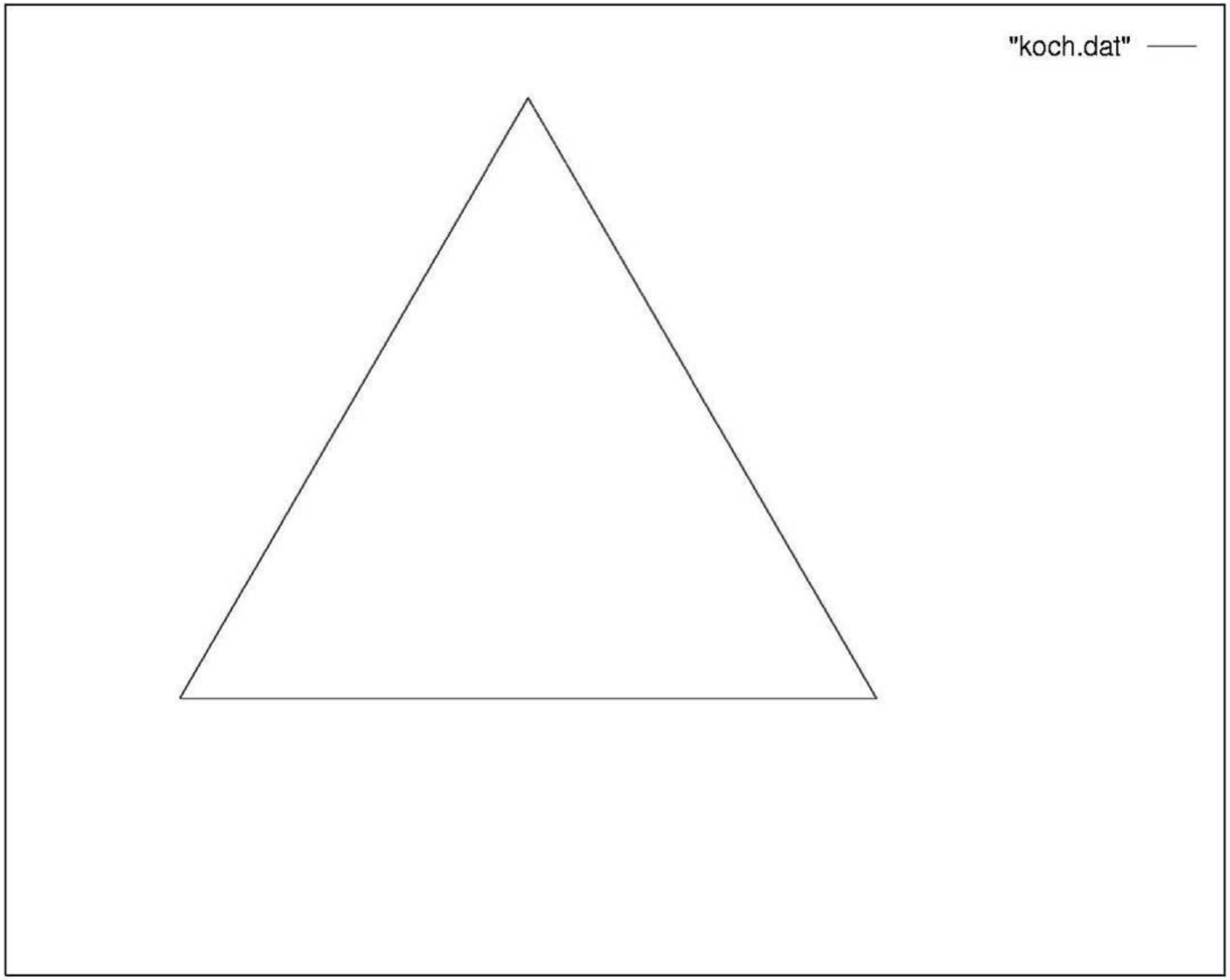
Sierpinski Triangle, 1916

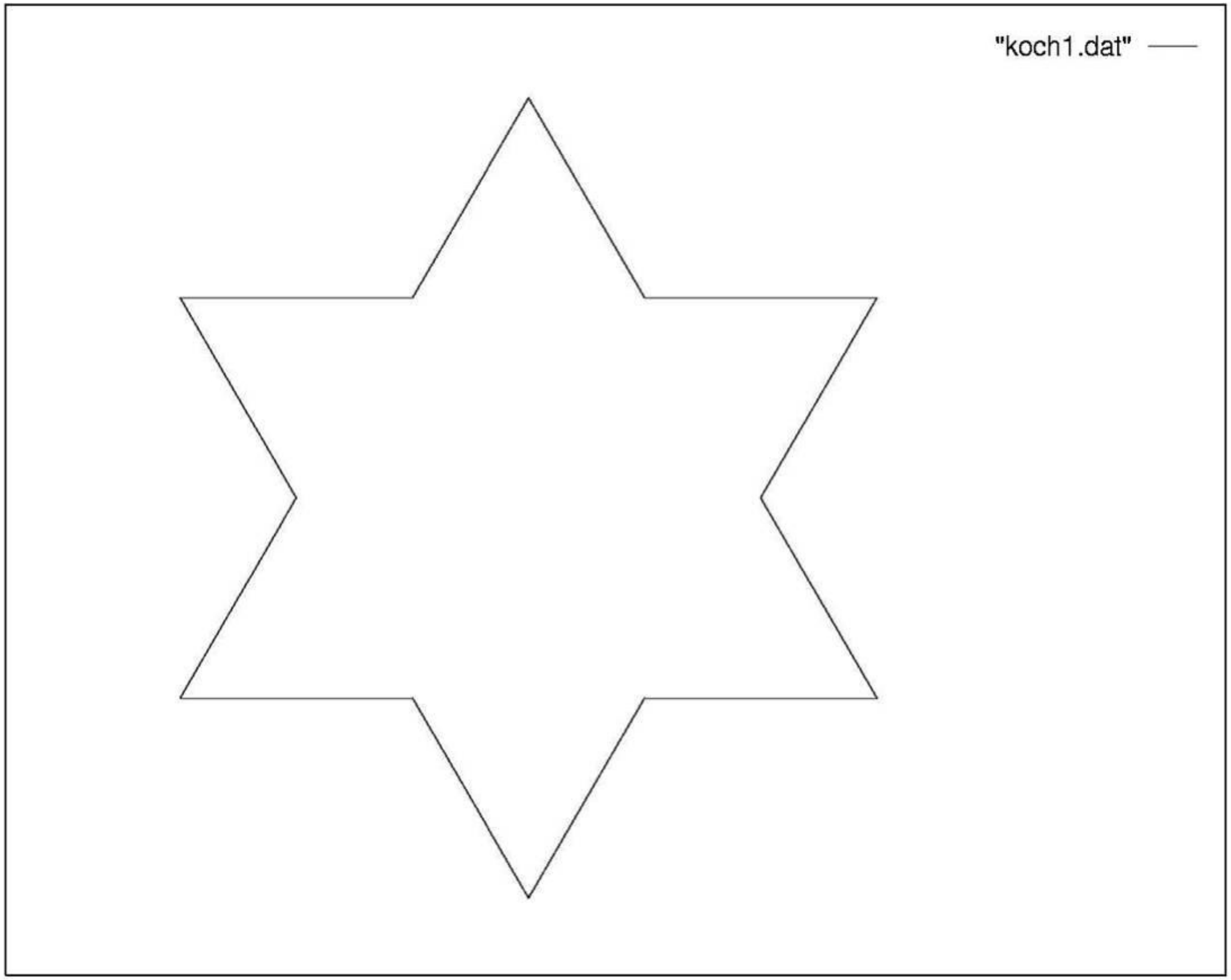
$D = Lim(\log(N_r)/\log(1/r))$

$= \log(2) / \log(3)$

r	N_r
1	1
1/3	2
1/3^2	2^2
1/3^3	2^3
1/3^n	2^n

- The Koch snowflake: We start with an equilateral triangle. We duplicate the middle third of each side, forming a smaller equilateral triangle. We repeat the process.


"koch.dat"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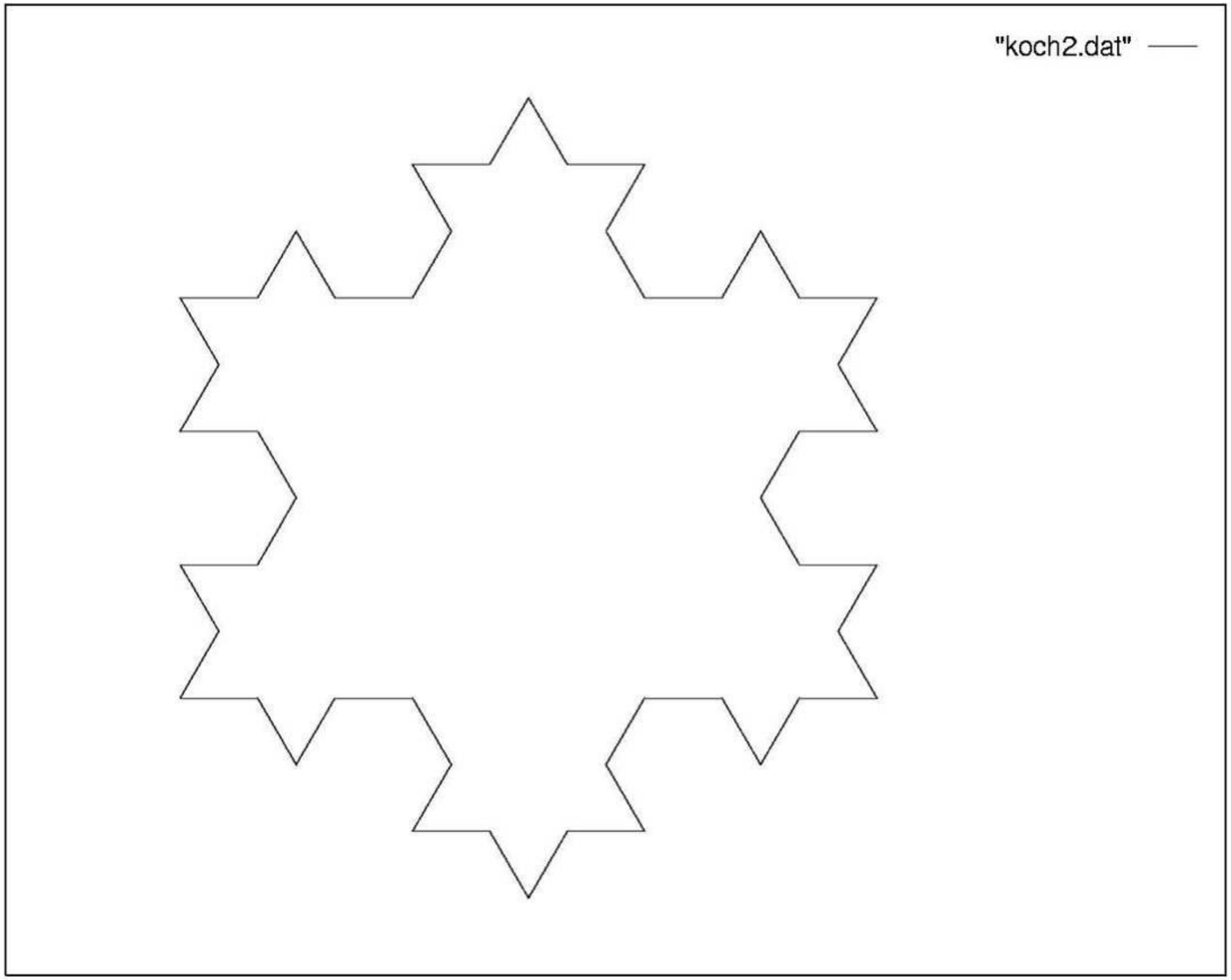
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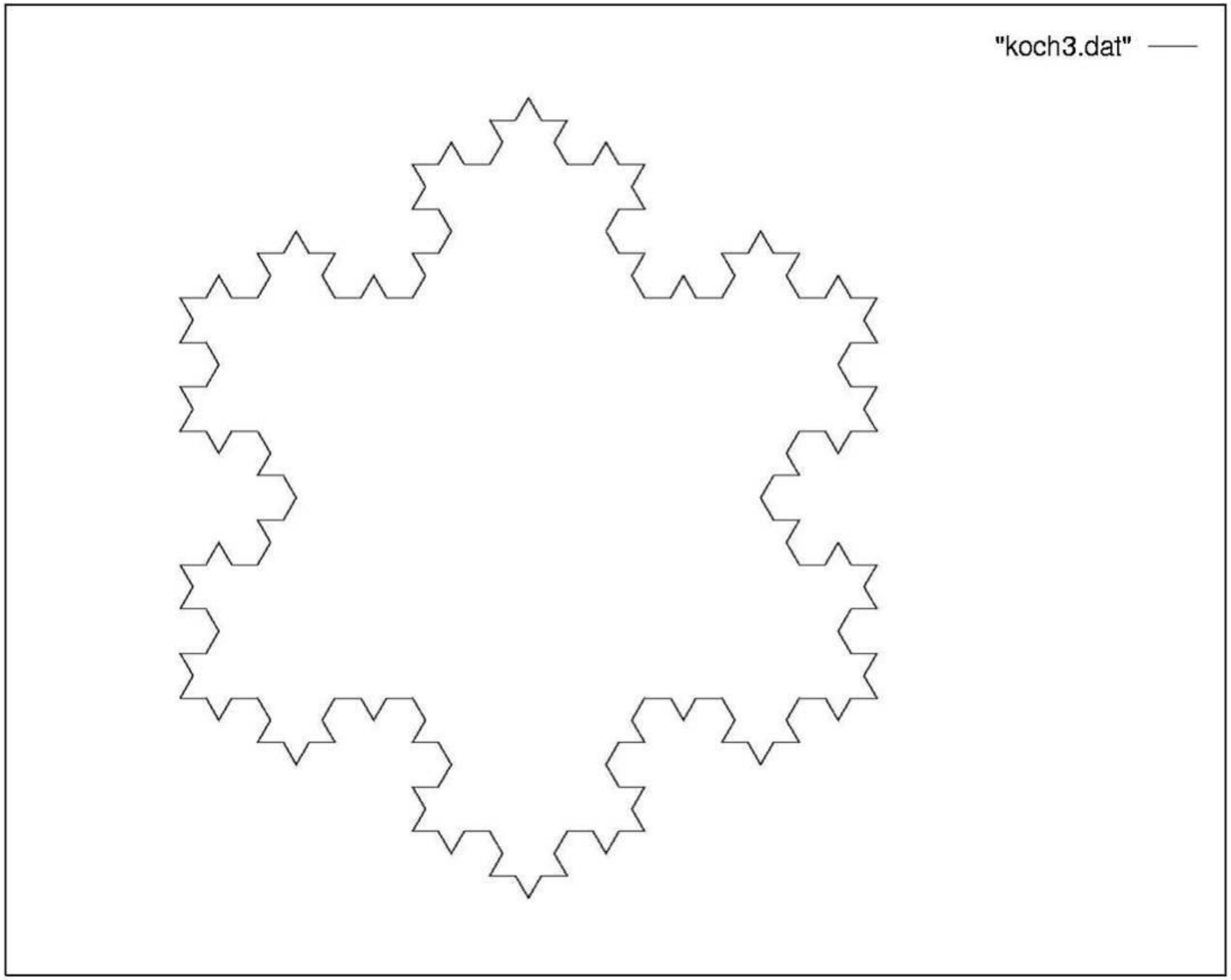
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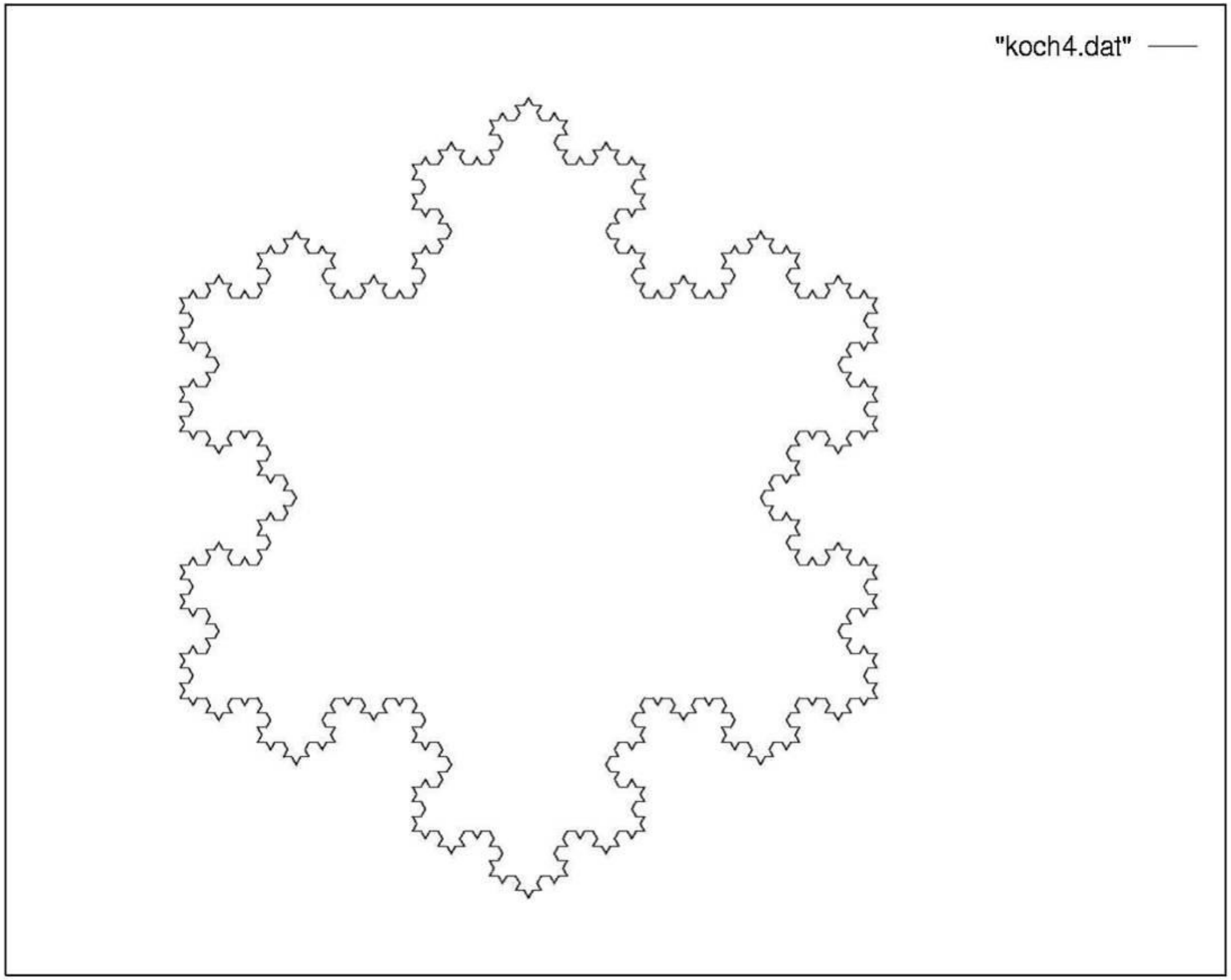
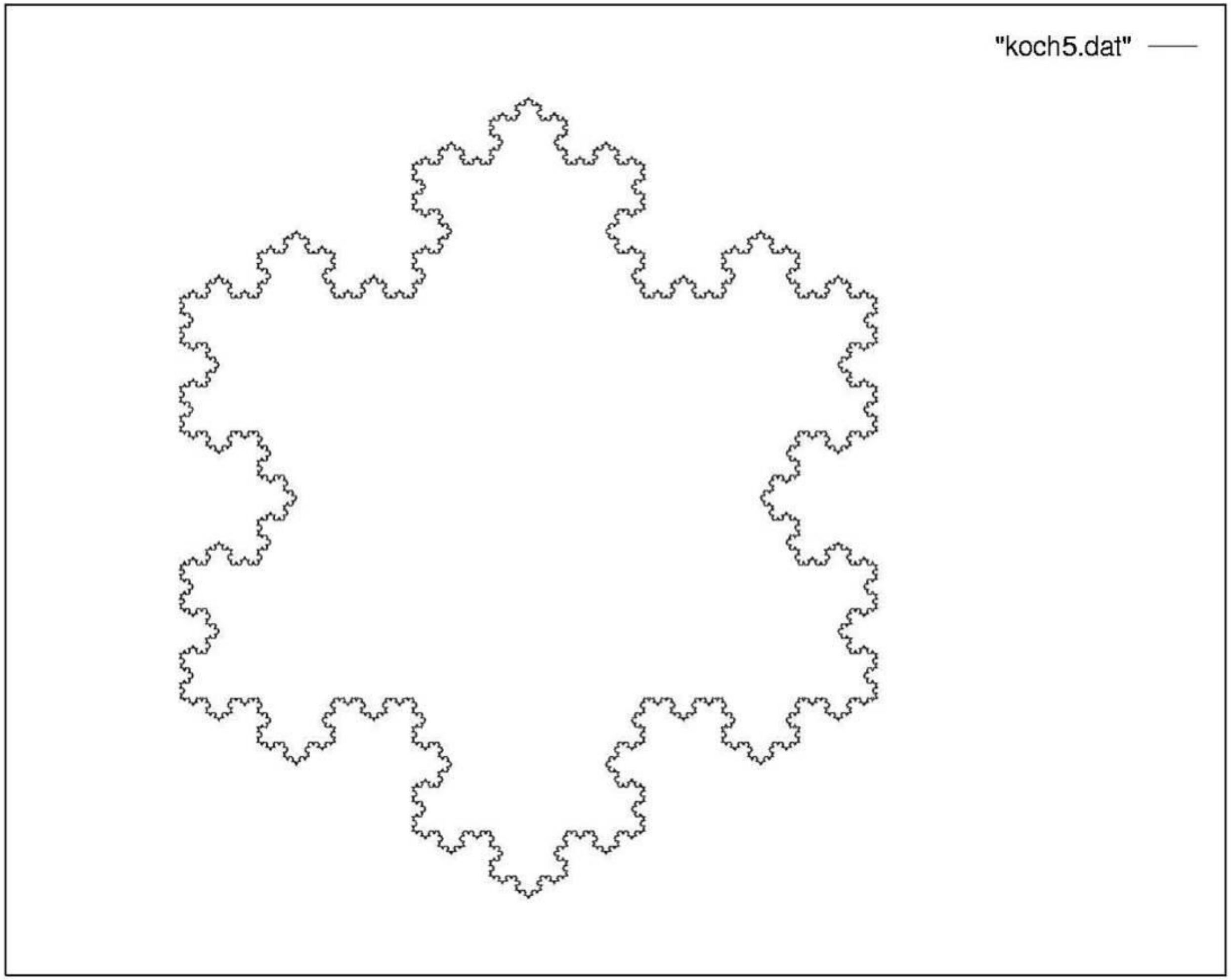

"koch4.dat"


"koch5.dat"

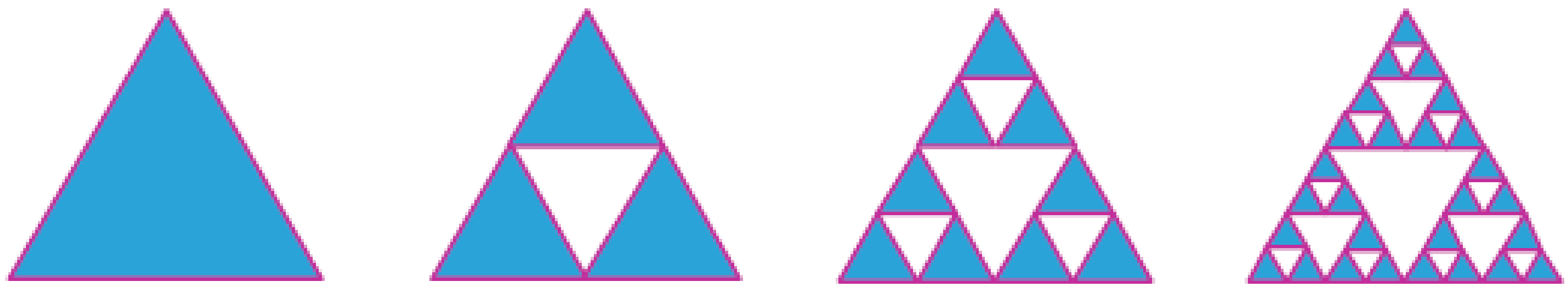
$D = \text{Lim}(\log(N_r)/\log(1/r))$

$= \log(3) / \log(2)$

r	N_r
1	1
1/2	3
1/2^2	3^2
1/2^3	3^3
1/2^n	3^n

The Sierpinsky Gasket

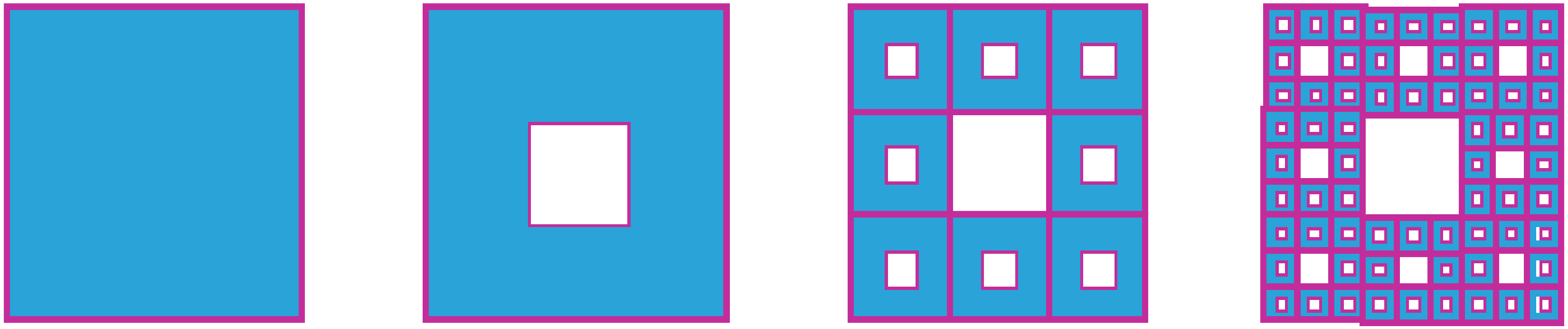
- Waclaw Sierpinsky, 1916



Initiator X_0 Generator X_1 X_2 X_3

- Dimension: log3k /log2k = log3/log2 ≈ 1.5849
- The limit object consists of branching points

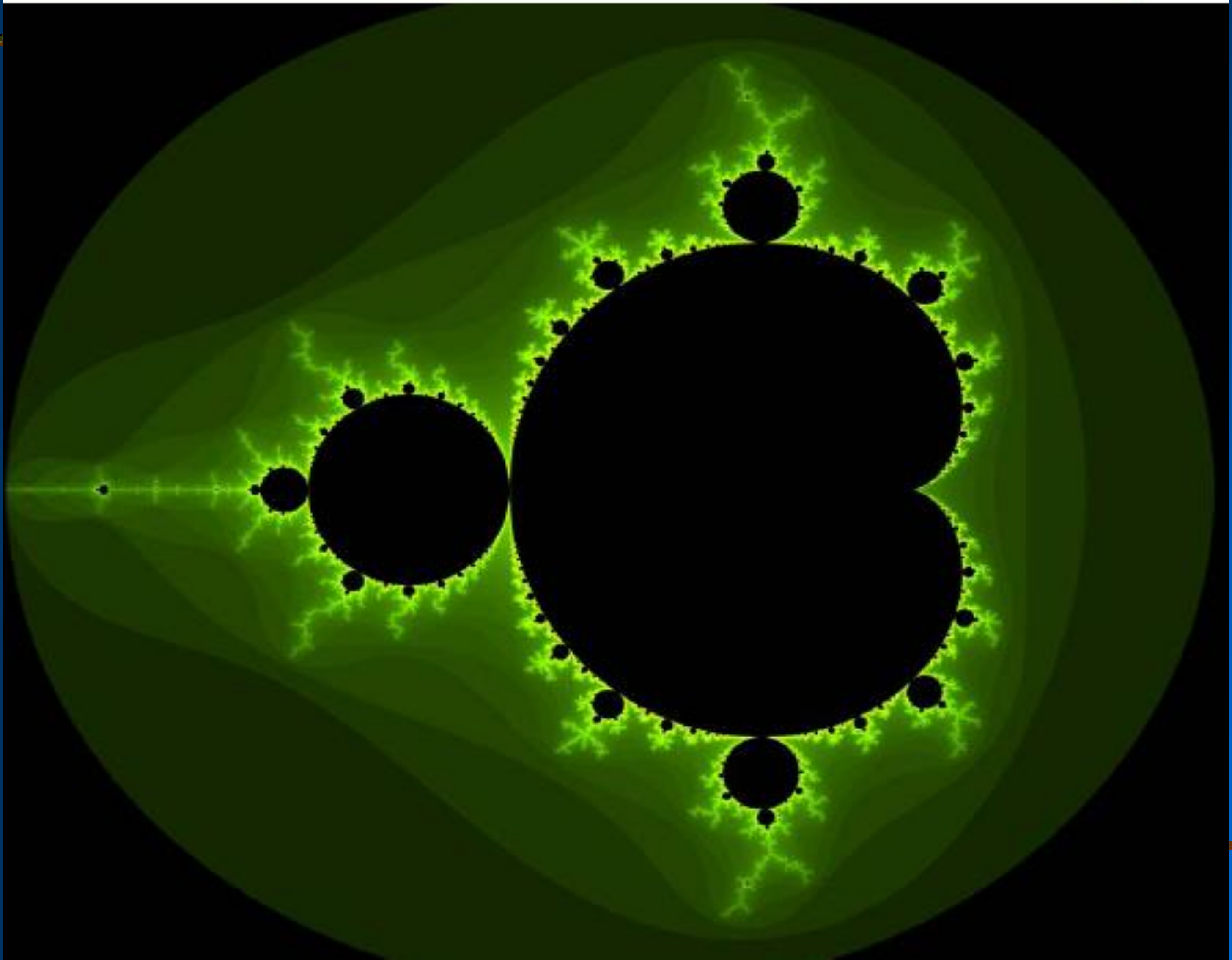
The Sierpinsky Carpet



Initiator X_0 Generator X_1 X_2 X_3

- Dimension: $\log 8^k / \log 3^k = \log 8 / \log 3 \approx 1.892$
- The limit object consists of branching points
- Univeral: it contains a topological version of any 1-dimensional object

Mandelbrot Set



Self Similarity of Fractals

- Line: $\log 3/\log 3 = 1$
- Square: $\log 9/\log 3 = 2$
- Cube: $\log 27/\log 3 = 3$

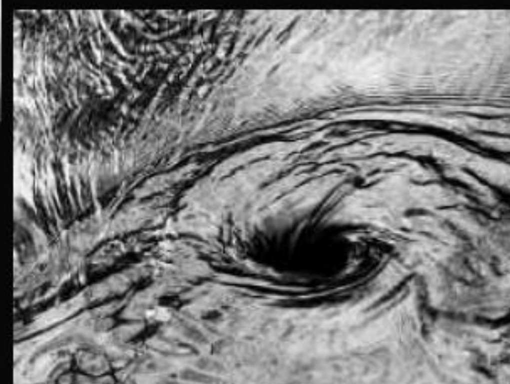
- Cantor Set:

  $\log 2^k / \log 3^k = \log 2/\log 3 \approx 0.6309$
- Koch Curve:

  $\log 4^k / \log 3^k = \log 4/\log 3 \approx 1.2619$

Dynamics

"All is process. That is to say, there is 'no thing' in the universe. Things, objects, entities, are abstractions of what is relatively constant from a process of movement and transformation. They are like the shapes that children like to see in clouds." — David Bohm, *Physicist* (1917-1992)



Is Your World Linear or Nonlinear?

Linear

- Simple rules ⟶ simple behaviors
- Things add up (superposition)
- Proportionality of input/output
- *High predictability, no surprises*

Is Your World Linear or Nonlinear?

Non-Linear

- Simple rules → complex behaviors
- *Small changes may have huge effects*
- Low predictability & anomalous behaviors
- Whole ≠ sum of parts ("emergent" properties)

Wonderful World of “ Hidden Complexity”

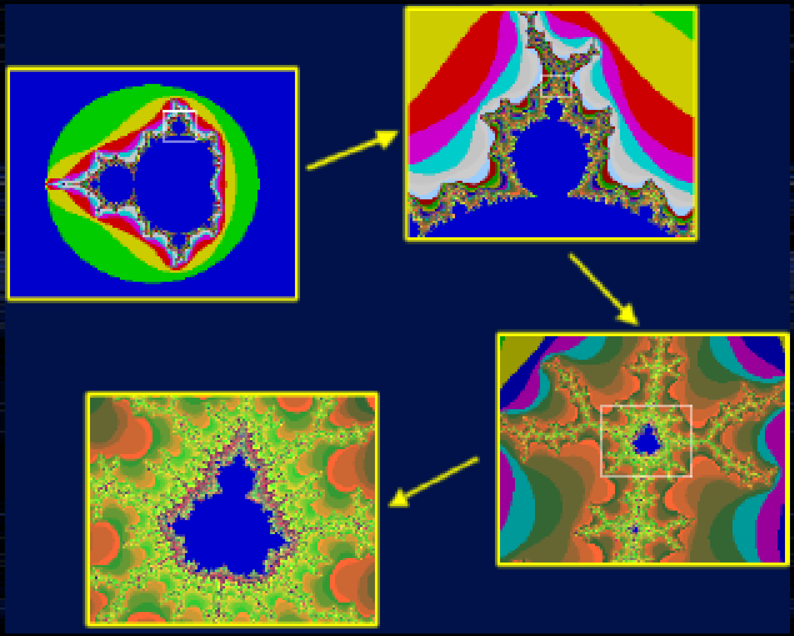
Bifurcations

Nonlinear oscillations

Deterministic chaos

Time asymmetry

Fractals



- Nonlinear waves: spirals/scrolls/solitons
- Stochastic resonance
- Complex networks
- Hysteresis
- Emergence

Chaos

"Chaos is a name for any order that produces confusion in our minds." — G. Santayana, *Philosopher* (1863-1952)

The study of how simple — but *nonlinear* — systems can generate complicated dynamics

$$f(x + y) \neq f(x) + f(y)$$

Deterministic Chaos

Irregular or random appearing motion in nonlinear dynamical systems whose dynamical laws uniquely determine the time evolution of the state of the system from a knowledge of its past history.

- Chaos is not due to
  - External noise
  - Having an infinite number of degrees-of-freedom
  - Quantum mechanical uncertainty

- Chaos is due to an *intrinsic* sensitivity to initial conditions

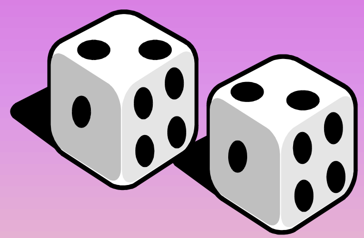
Characteristics of Systems

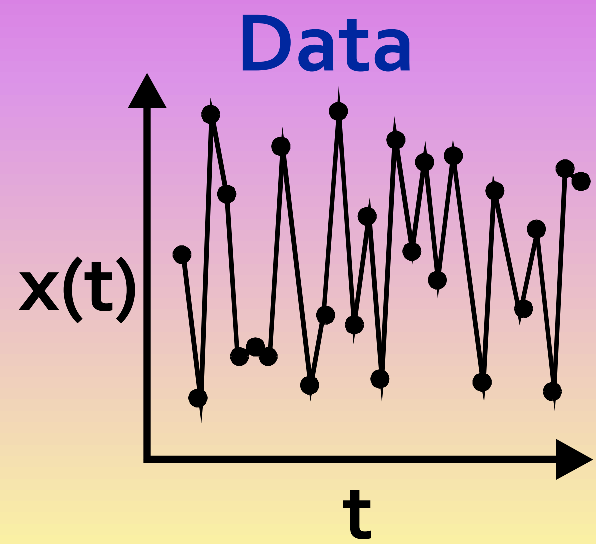
System	Order	Chaos	Randomnes s
Paradigmatic Example	Clocks, Planets	Clouds, Weather	Snow on TV Screen
Predictability	Very High	Finite, Short Term	None -> Simple Laws
Effect of Small Errors	Very Small	Explosive	Nothing BUT Errors
Spectrum	Pure	Yes!	Noisy, Broad
Dimension	Finite	Low	Infinite
Control	Easy	Tricky, Very Effective	Poor
Attractor	Point, Cycle,	Strange, Fractal	No!

Mechanism that Generated the Data

Chance

d(phase space set) $\longrightarrow \infty$

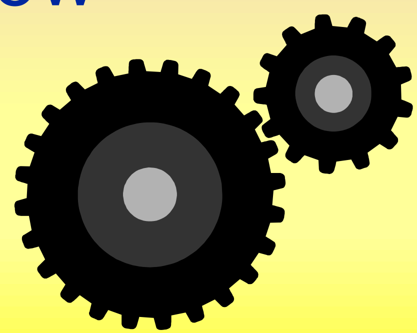




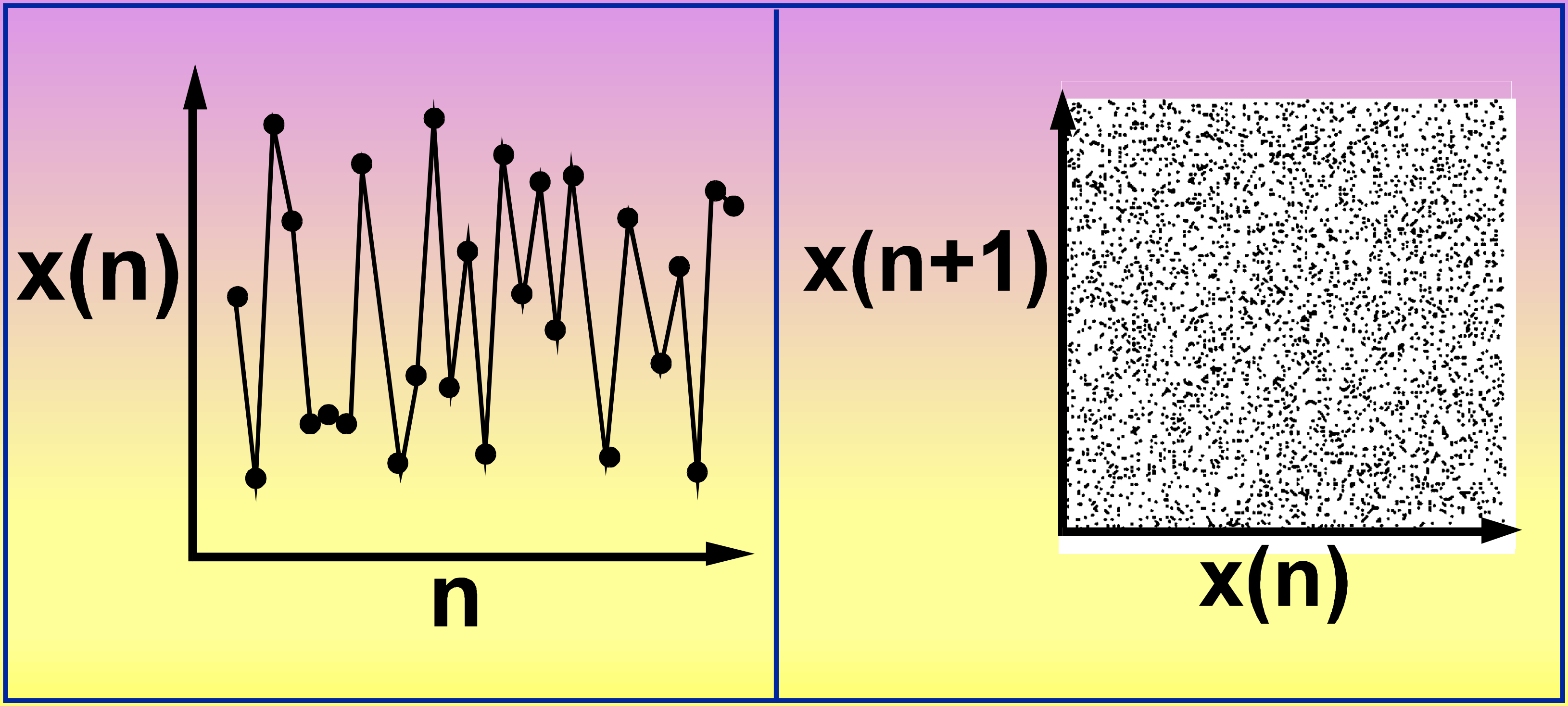
?

Determinism

d(phase space set)
= low



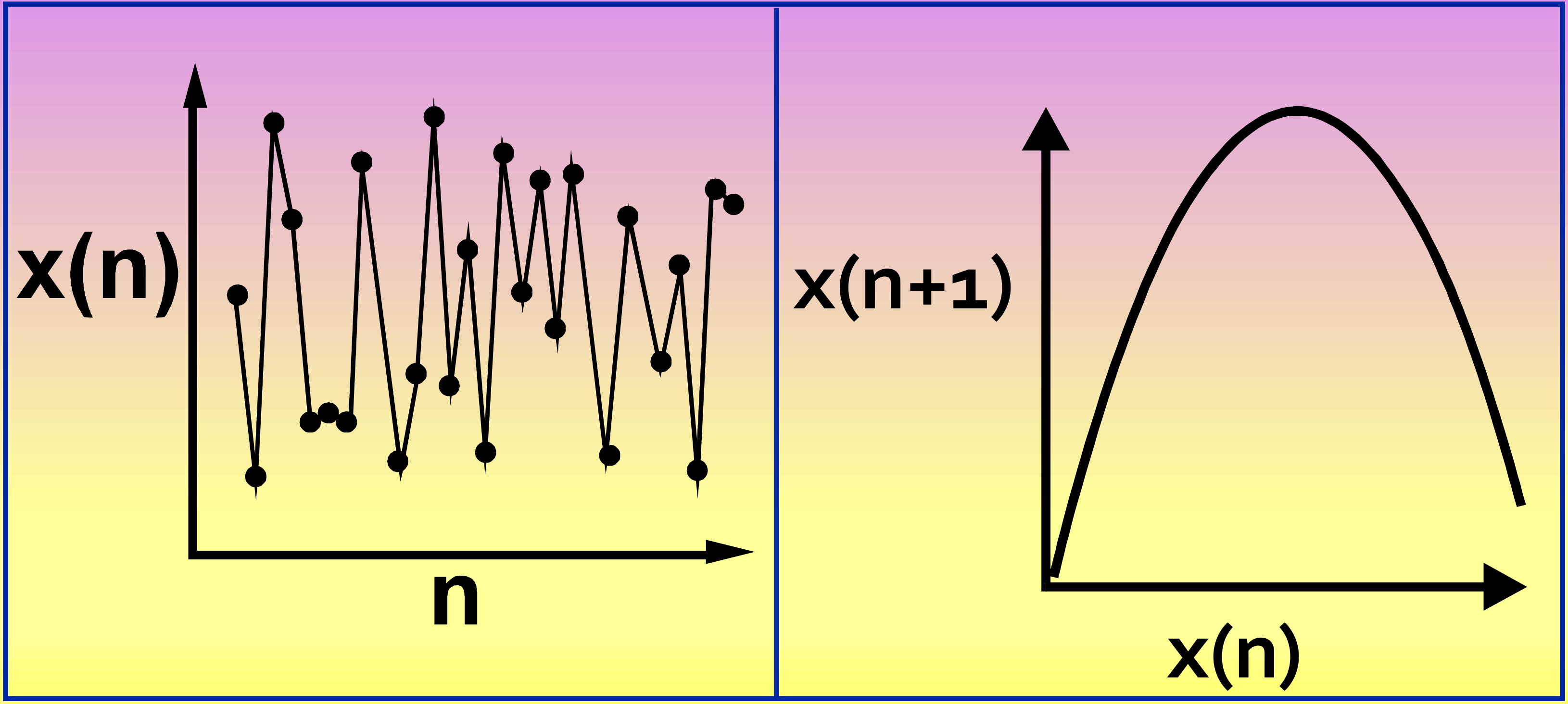
RANDOM $x(n) = RND$



CHAOS

deterministic

$$x(n+1) = 3.95\, x(n)\, [1 - x(n)]$$



Chaos vs. Randomness

- Do not confuse chaotic with random:

Random:

- irreproducible and unpredictable

Chaotic:

- deterministic - same initial conditions lead to same final state… **but the final state is very different for small changes to initial conditions**
- ***difficult or impossible to make long-term predictions***

CHAOS

Definition

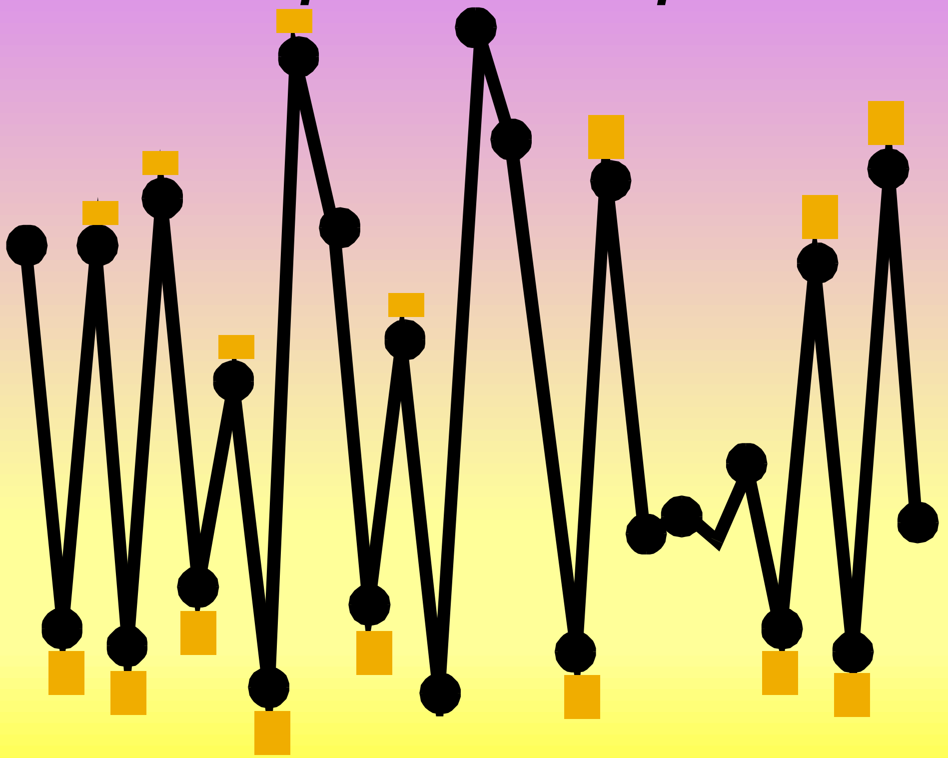
Small Number of Variables

x(n+1) = f(x(n), x(n-1), x(n-2))

CHAOS

Definition

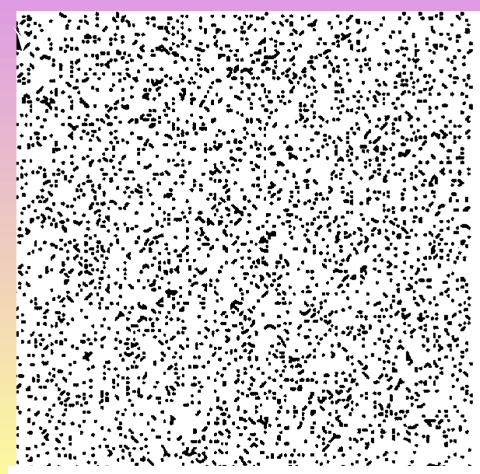
Complex Output

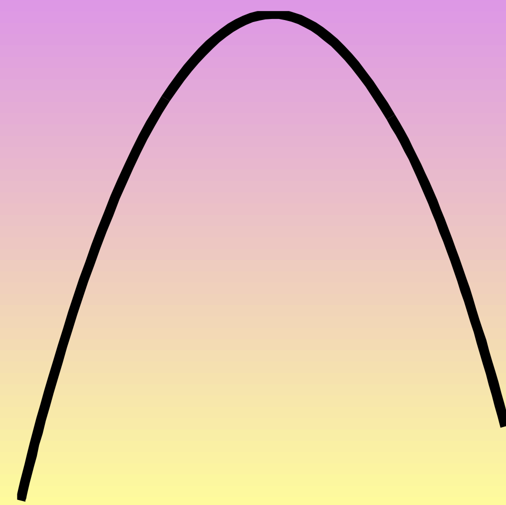


CHAOS

Properties

Phase Space is Low Dimensional





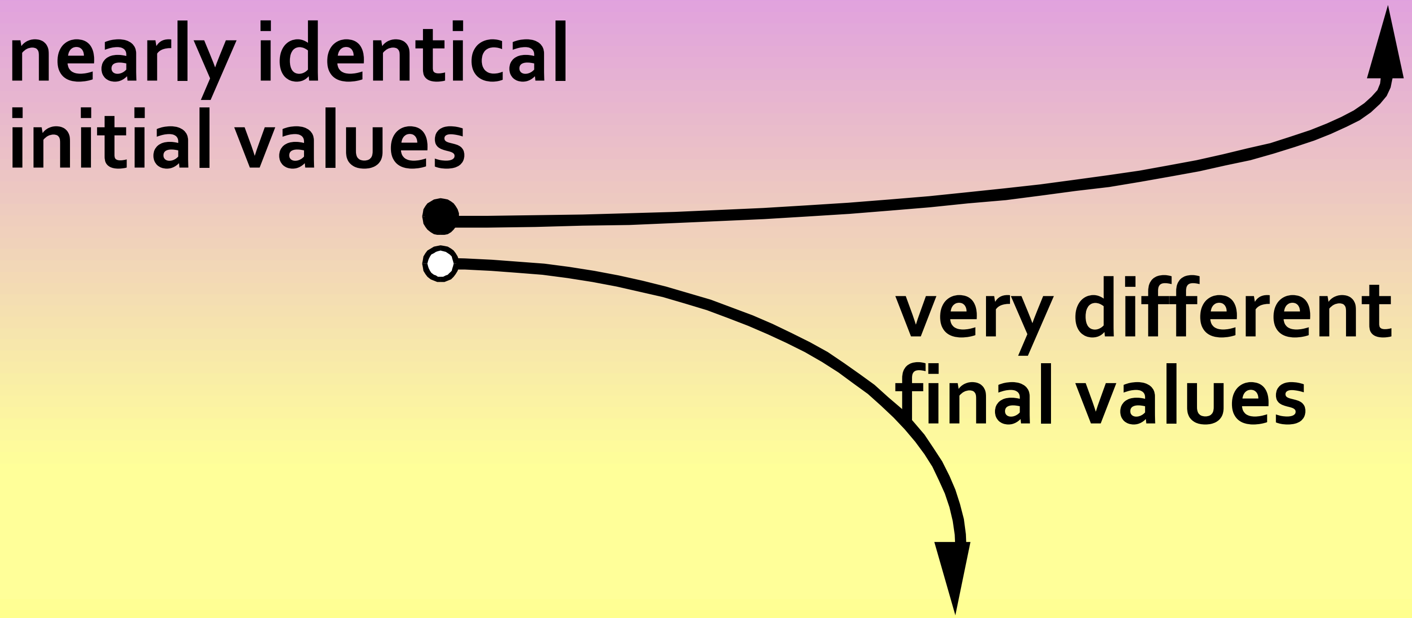
D ⟶ ∞ random | d = 1, chaos

phase space

CHAOS

Properties

Sensitivity to Initial Conditions



Lorenz

1963 J. Atmos. Sci. 20:13-141

Equations

$$\frac{dX}{dt} = 10\,(Y - X)$$

$$\frac{dY}{dt} = -XZ + 28X - Y$$

$$\frac{dZ}{dt} = XY - (8/3)Z$$

Lorenz

1963 J. Atmos. Sci. 20:13-141

Equations

X = speed of the convective circulation
X > 0 clockwise,
X < 0 counterclockwise

Y = temperature difference between rising and falling fluid

Lorenz

1963 J. Atmos. Sci. 20:13-141

Equations

Z = bottom to top temperature minus the linear gradient

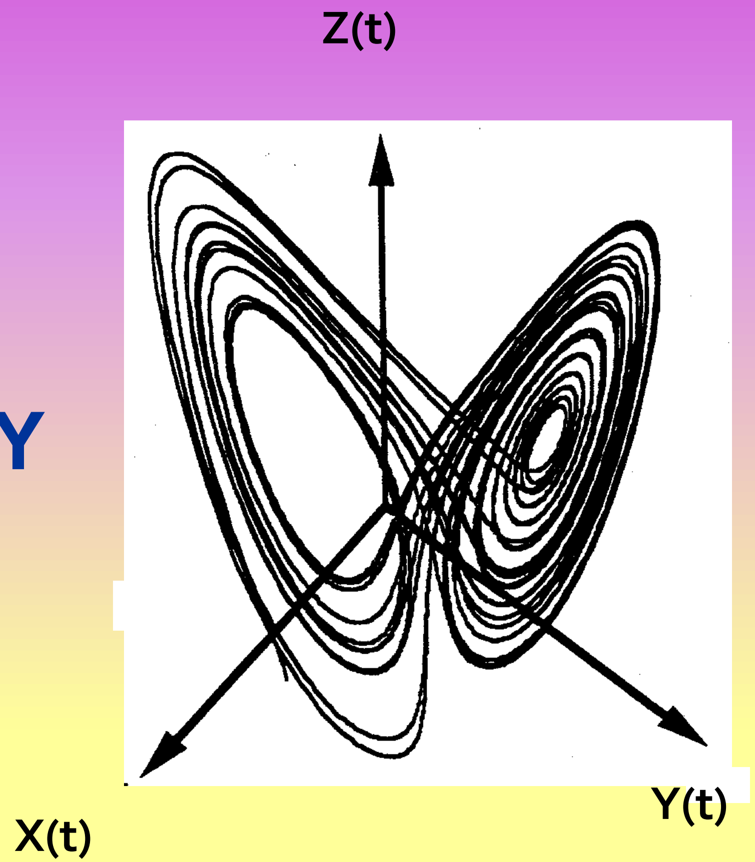
Attractors in Phase Space

Lorenz Equations

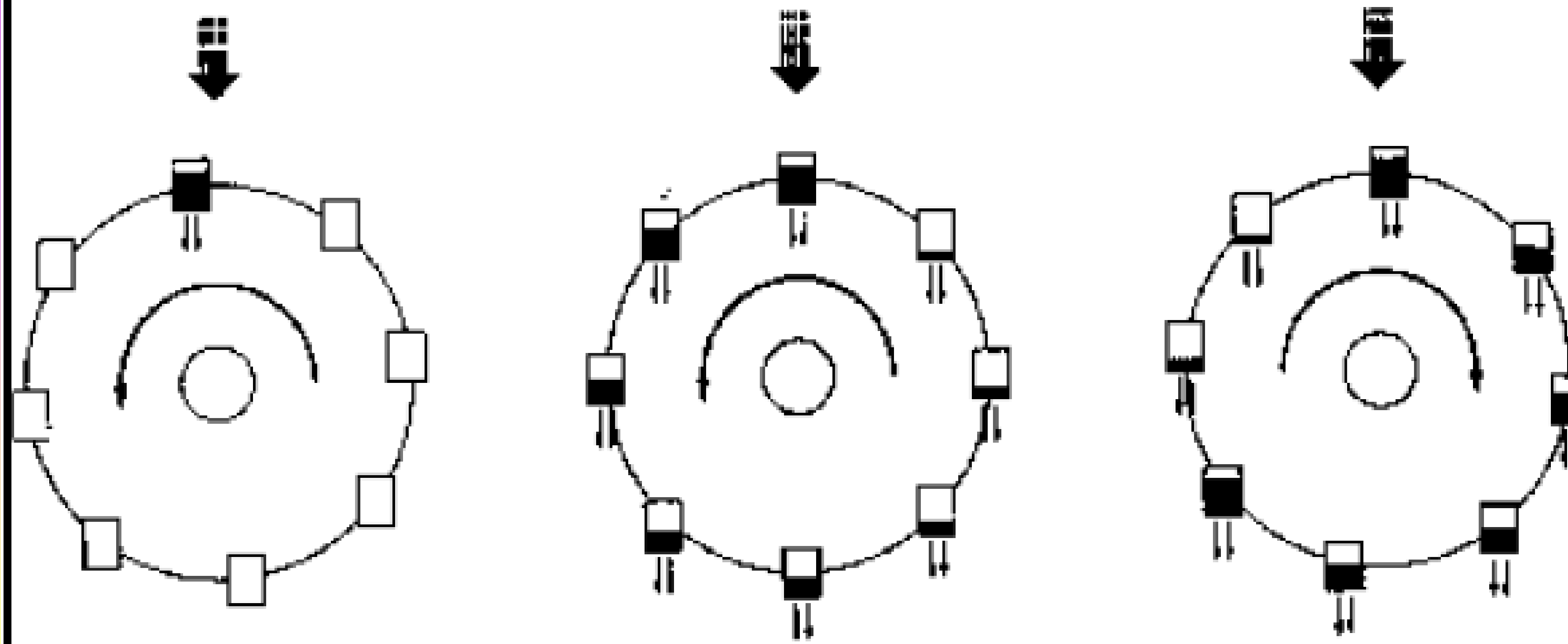
$$\frac{dX}{dt} = 10\,(Y - X)$$

$$\frac{dY}{dt} = -XZ + 28X - Y$$

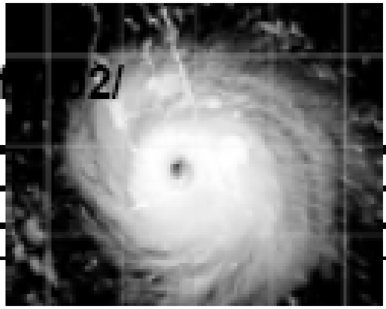
$$\frac{dZ}{dt} = XY - (8/3)Z$$



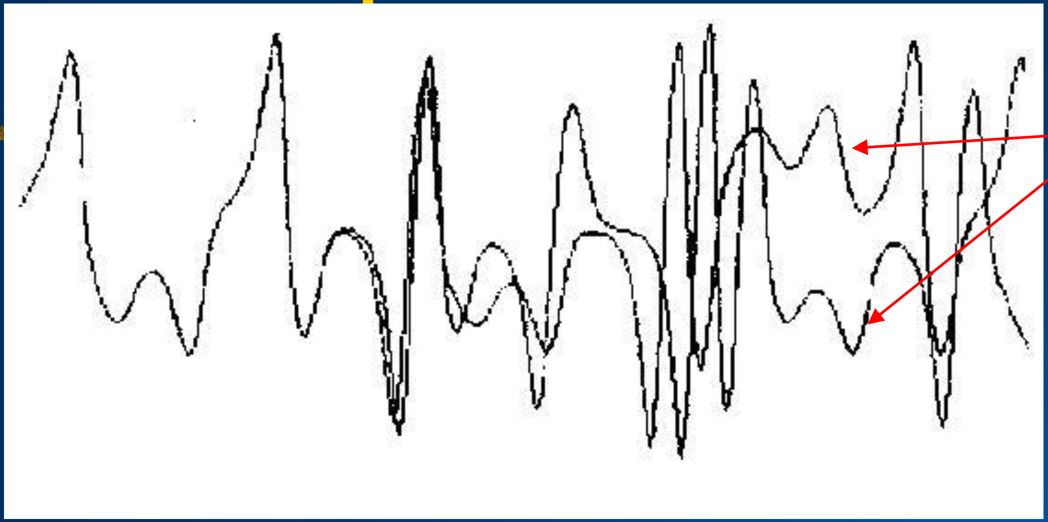
Lorenz Water Wheel



Source: http://www.ace.gatech.edu/experiments2/2413/lorenz/f 2/



Examples of CHAOS

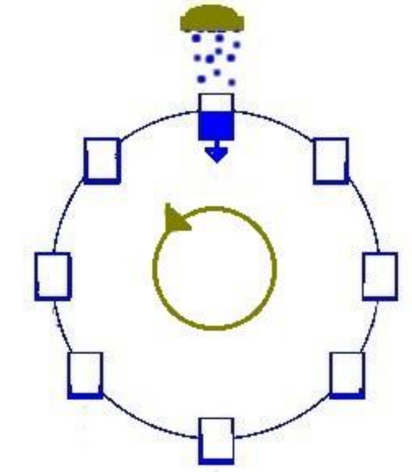


Divergent behavior with almost the same set of initial conditions

1. Solution of Lorenz's weather prediction model.

2. Waterwheel experiment (Lorenz's waterwheel)

- mathematical model : serendipity - Lorenz



As water flow increases, enough water collects in the top cup to start the wheel turning either to the right or left.

Edward Lorenz and the Butterfly Effect (cont.)

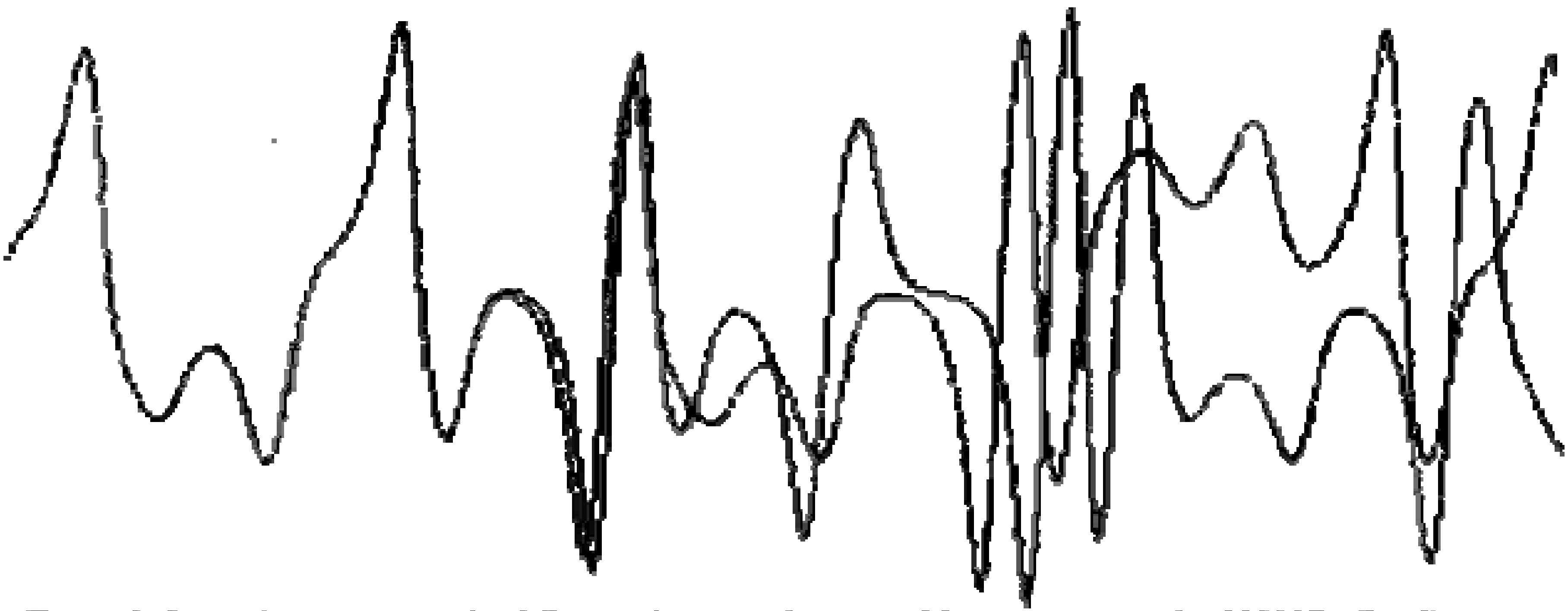


Figure 1: Lorenz's experiment: the difference between the start of these curves is only .000127. (Ian Stewart, Does God Play Dice? The Mathematics of Chaos, pg. 141)

Characteristics of Chaos

- **Deterministic** – Chaotic systems are completely deterministic and not random.
- **Sensitive** – Chaotic systems are extremely sensitive to initial conditions, since any perturbation, no matter how minute, will forever alter the future of the chaotic system.
- **Ergodic** – Chaotic motion is ergodic, which means that the state space trajectory of a chaotic system will always return to the local region of a previous point in the trajectory.
- **Embedded** – Chaotic attractors are embedded with an infinite number of unstable periodic orbits.

Edward Lorenz and the Butterfly Effect (cont.)

- This effect came to be known as the butterfly effect.

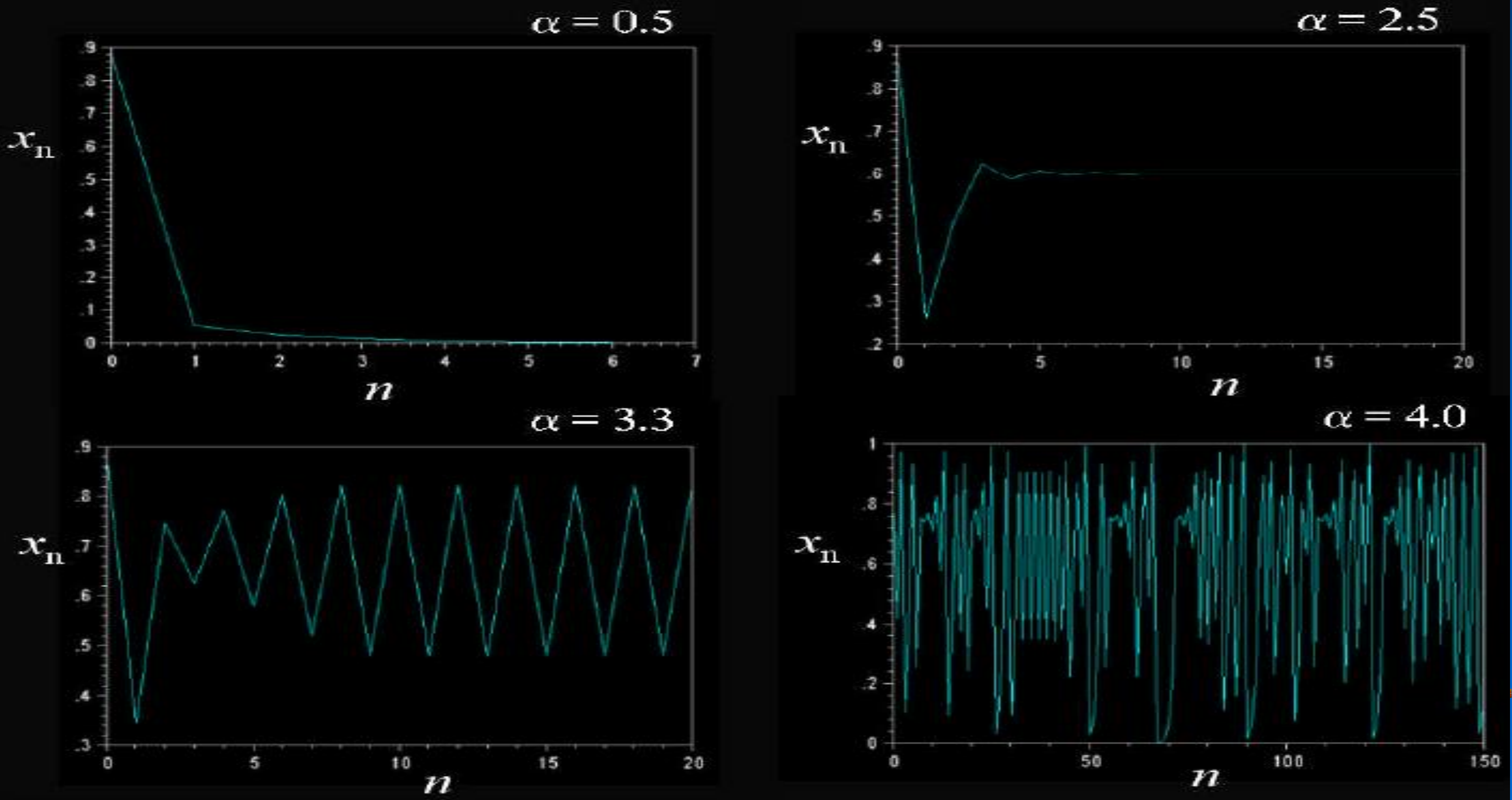
  "The flapping of a single butterfly's wing today produces a tiny change in the state of the atmosphere. Over a period of time, what the atmosphere actually does diverges from what it would have done. So, in a month's time, a tornado that would have devastated the Indonesian coast doesn't happen. Or maybe one that wasn't going to happen, does." [Ian Stewart, *Does God Play Dice? The Mathematics of Chaos*]

- This phenomenon is also known as sensitive dependence on initial conditions.

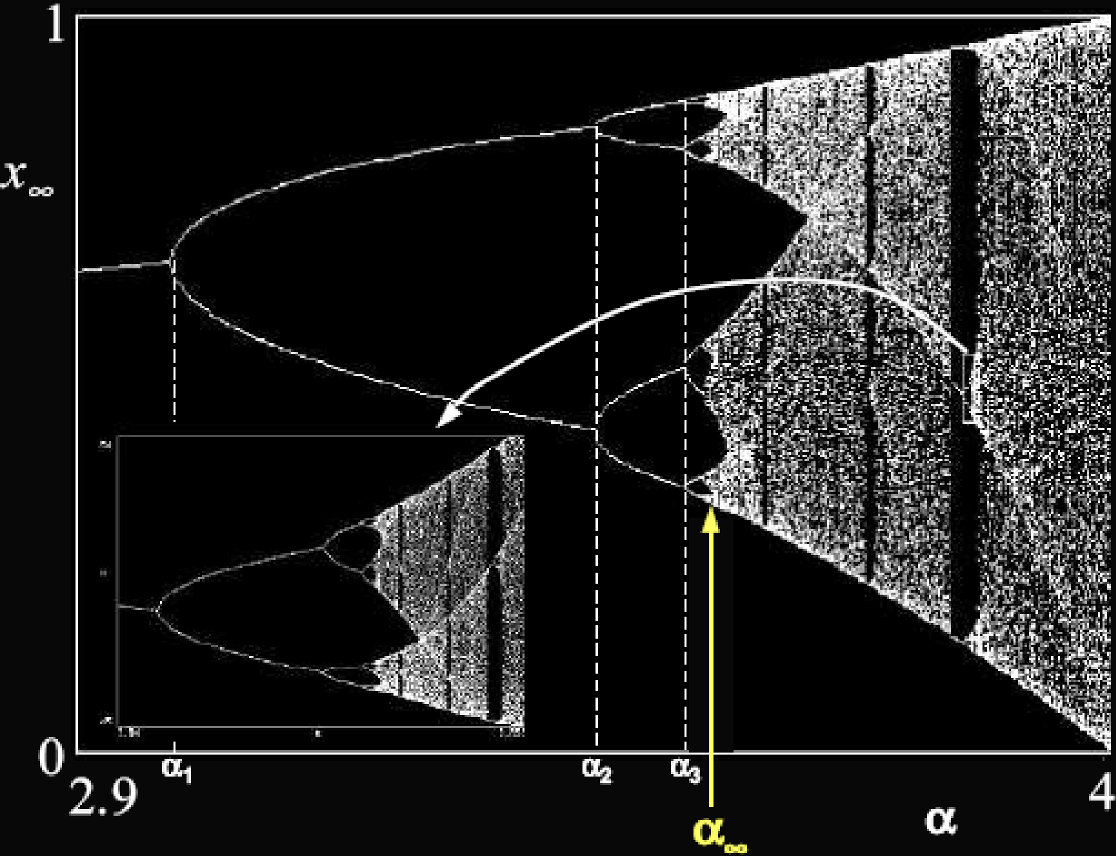
Logistic Equation

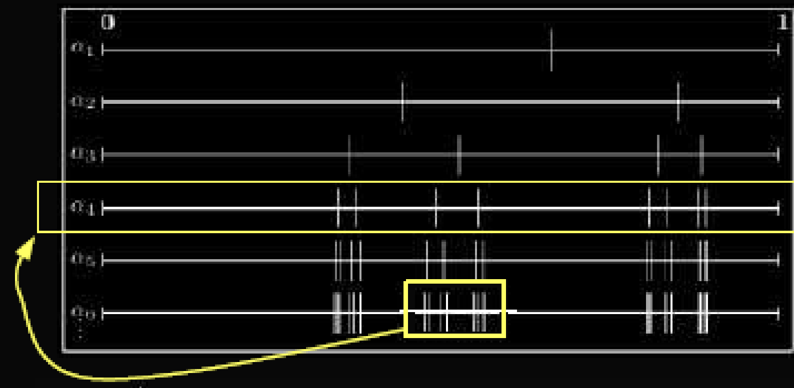
"Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realized that simple dynamical systems do not necessarily lead to simple dynamical behavior." — R. M. May, *Nature* (1976)

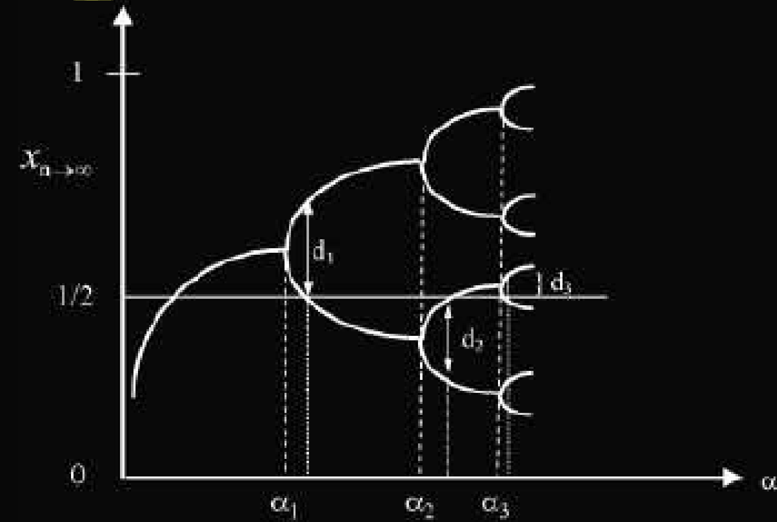
$$x_{n+1} = \alpha\, x_n\, (1 - x_n), \quad 0 \le \alpha \le 4, \quad 0 \le x_0 \le 1$$



What happens as a function of α?







What does the orbit look like for α_∞?

It is an infinite, self-similar set; i.e. a *Fractal!*

$$\begin{cases} \dfrac{\alpha_n - \alpha_{n-1}}{\alpha_{n+1} - \alpha_n} \longrightarrow \delta = 4.6692016... \\ \dfrac{d_n}{d_{n+1}} \longrightarrow \alpha = 2.5029078... \end{cases}$$

What if we consider $x_{n+1} = \alpha \sin(\pi x_n)$ instead of $x_{n+1} = \alpha\, x_n (1 - x_n)$?

Modeling complexity in physics (history)

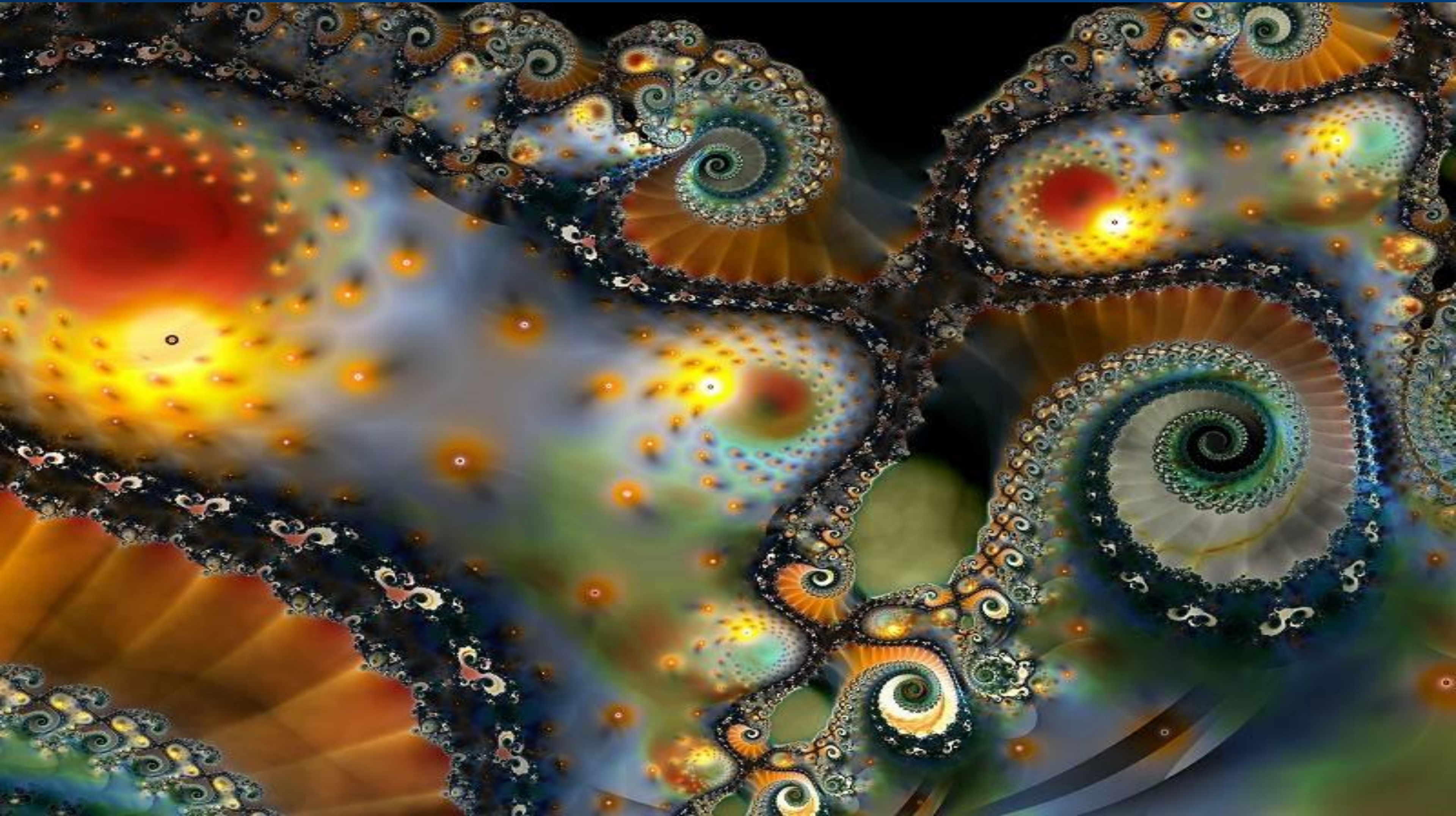
- **Newton**
  - Random walks '05
  - Classical diffusion '05
  - Langevin equation '16
  - Fokker-Planck equation '17
    - Gaussian statistics
    - Normal diffusion

$$\left\langle X(t)^2 \right\rangle \propto t$$

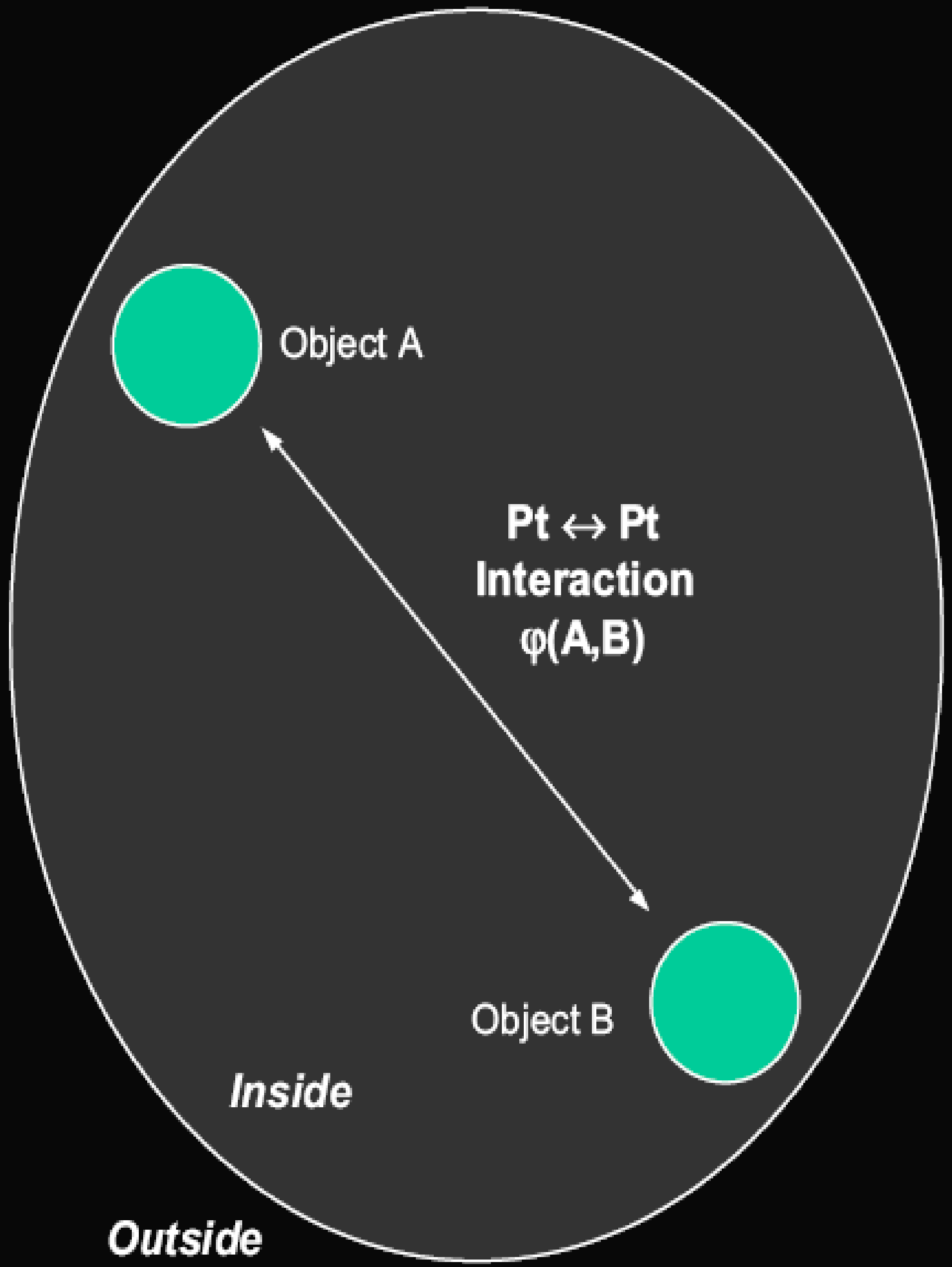
- **Mandelbrot**
  - Fractional Random walks '81
  - Anomalous diffusion '75
  - Fractional Langevin equation '63
  - Fractional diffusion equation '83
    - Lévy statistics
    - Anomalous diffusion

$$\left\langle X(t)^2 \right\rangle \propto t^{2H}$$

Thank you!



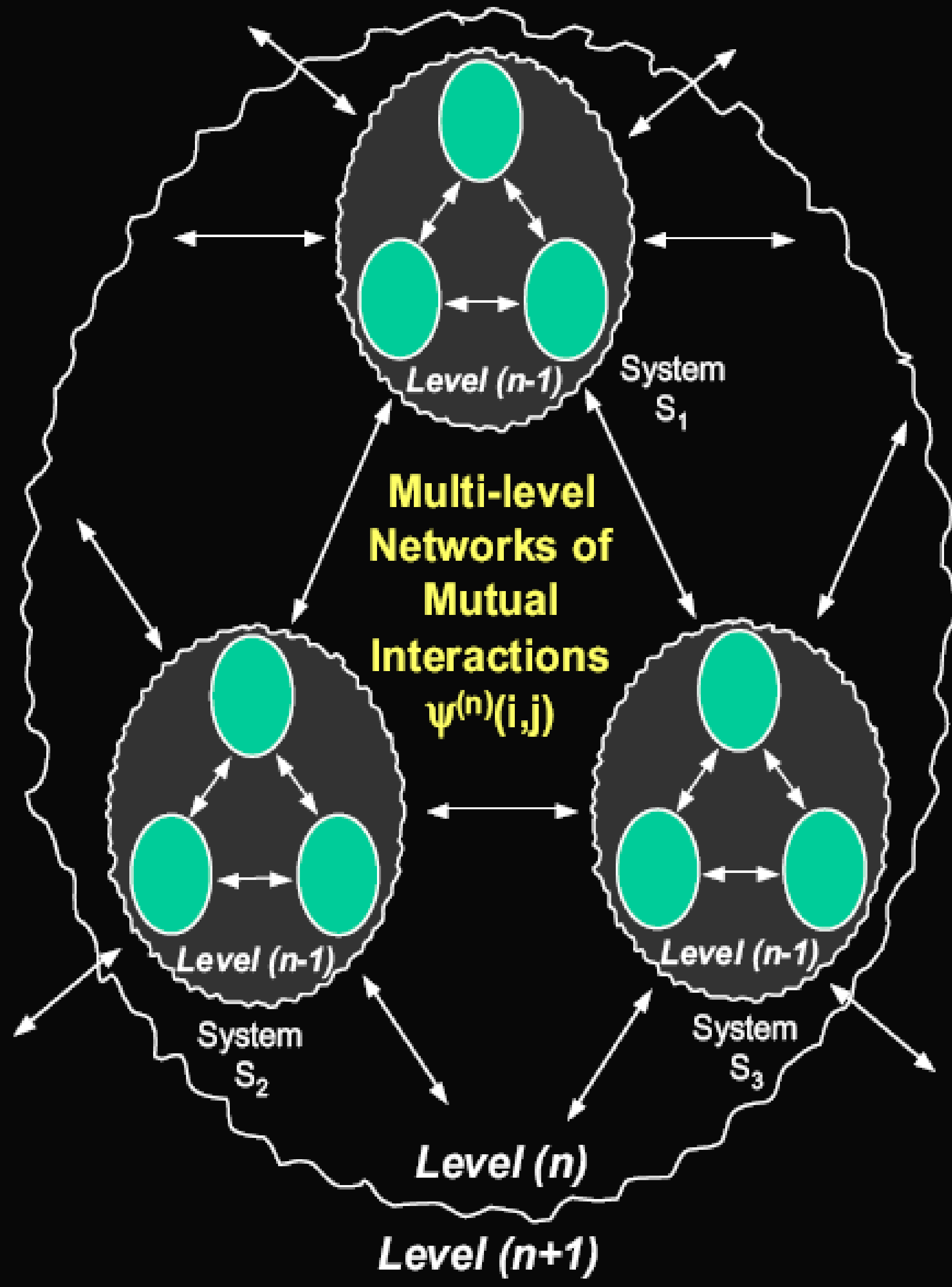
Complex Systems: A Gentle Introduction



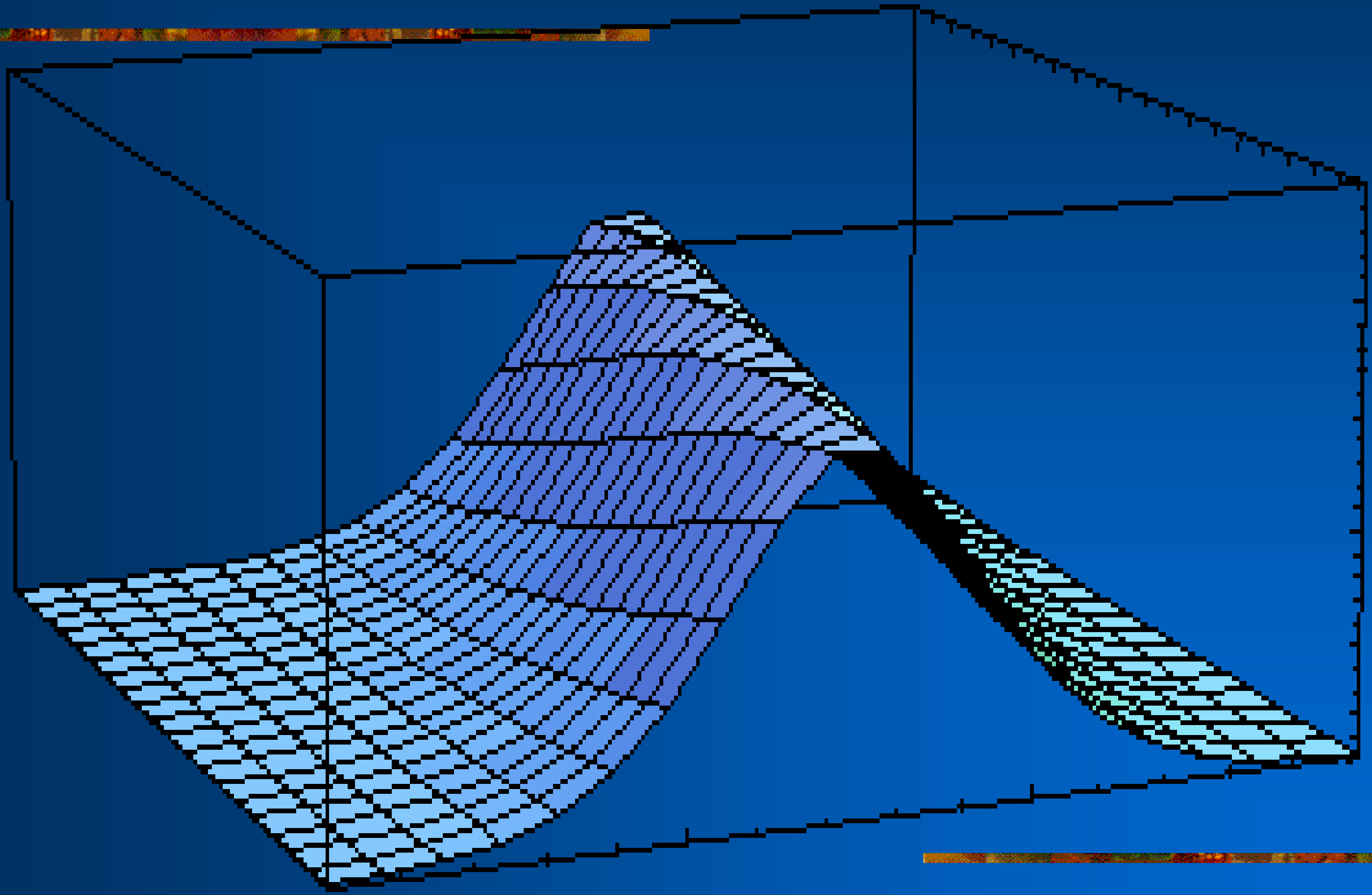
- Static
- Linear
- Homogeneous
- In Equilibrium
- Stable
- Predictable
- Reductionist
- “Closed System”
- Autonomous

Complex Systems: A Gentle Introduction

- Dynamic
- Nonlinear
- Heterogenous
- Far from Equilibrium
- Poised near Edge-of-Chaos
- Unpredictable
- Holistic
- "Open System"
- Interconnected
- *Universal Behaviors?*



Solitons



On shallow water solitary waves ...

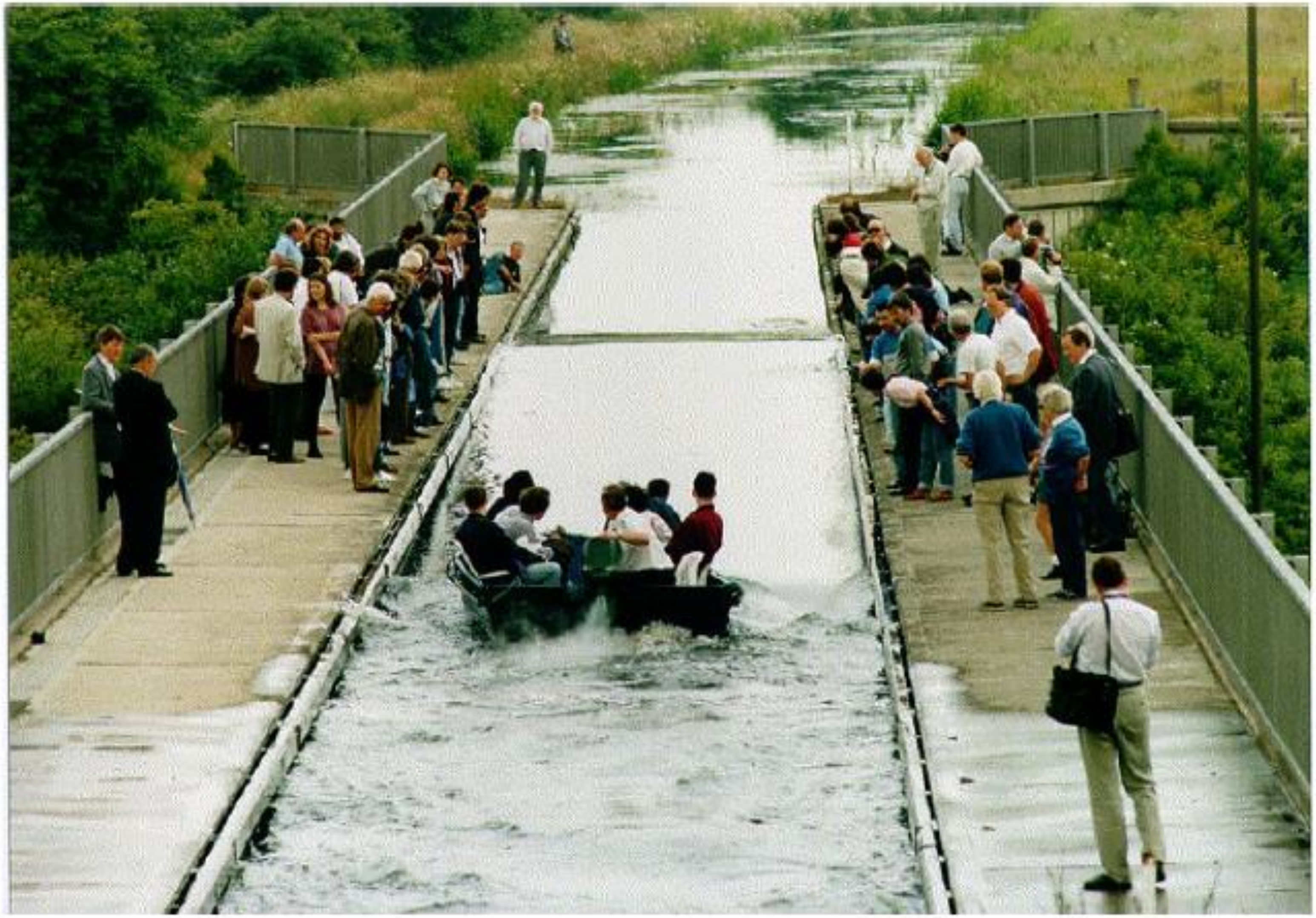
- 1834-1844: **John Scott Russell** experimentally observed his ***"great solitary wave of translation"*** in 1834 and reported it during the 1844 Meeting of the British Association for the advancement of science.

- It was not immediately believed that Scott Russell's solitary wave was of importance. Theoretical descriptions had to wait until ...

- 1871: French mathematician **Joseph Valentin Boussinesq** proposed his dispersive-nonlinear model for water surface waves, inspired by Russell's observation.

Joseph Valentin Boussinesq

Russell's soliton recreated in 1995



(Heriot-Watt University.)

- *In 1876*, Lord Rayleigh published his mathematical theory to support Russell's experimental observation.

- 1895: 20 years later, Dutch mathematician **Diederik Johannes Korteweg** and his doctoral student **Gustav de Vries**

Diederik Johannes Korteweg

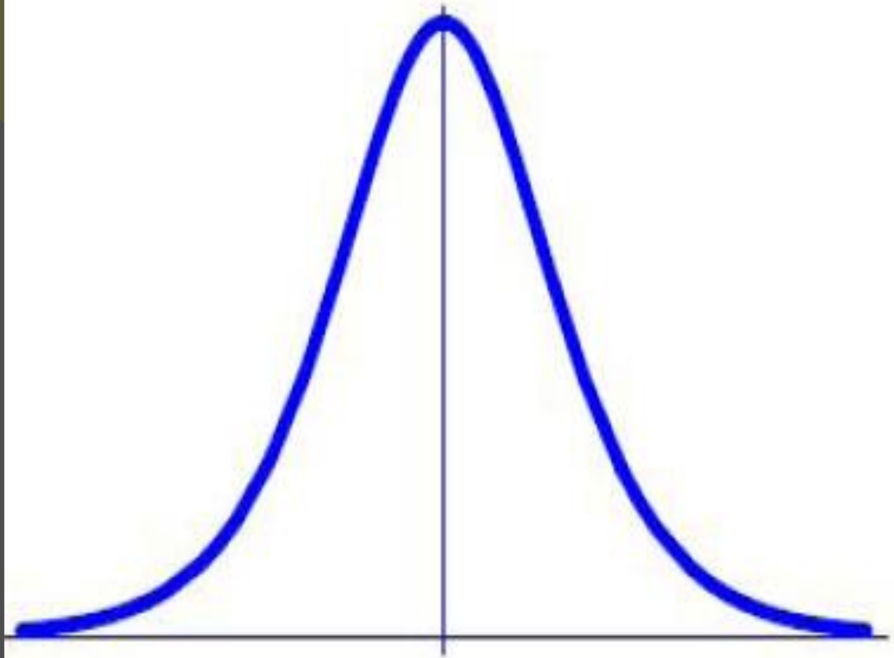
Gustav de Vries

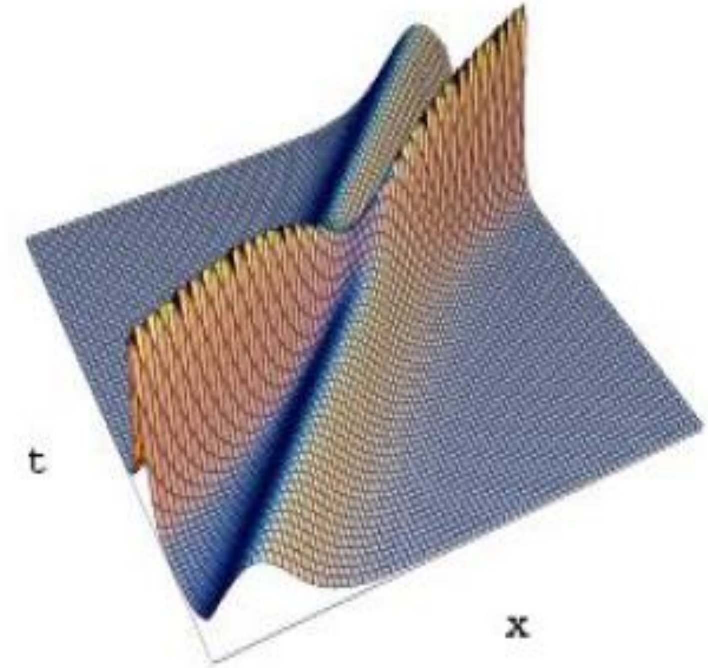
developed their theory for shallow water waves (*the KdV theory*).

The *Korteweg-de Vries (KdV) equation* reads:

$$\frac{\partial \phi}{\partial t} + a\,\phi\,\frac{\partial \phi}{\partial x} + b\,\frac{\partial^3 \phi}{\partial x^3} = 0$$

where a is the nonlinearity coefficient, and b is the dispersion coefficient.

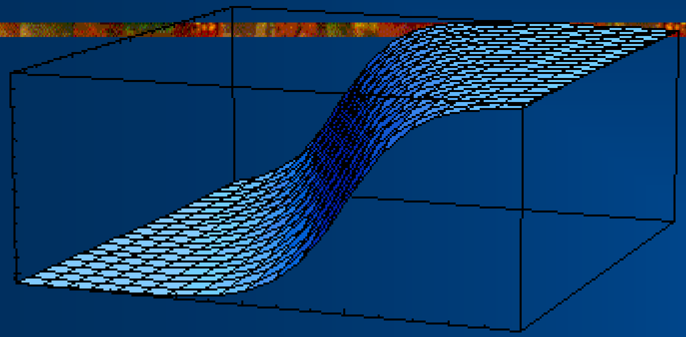




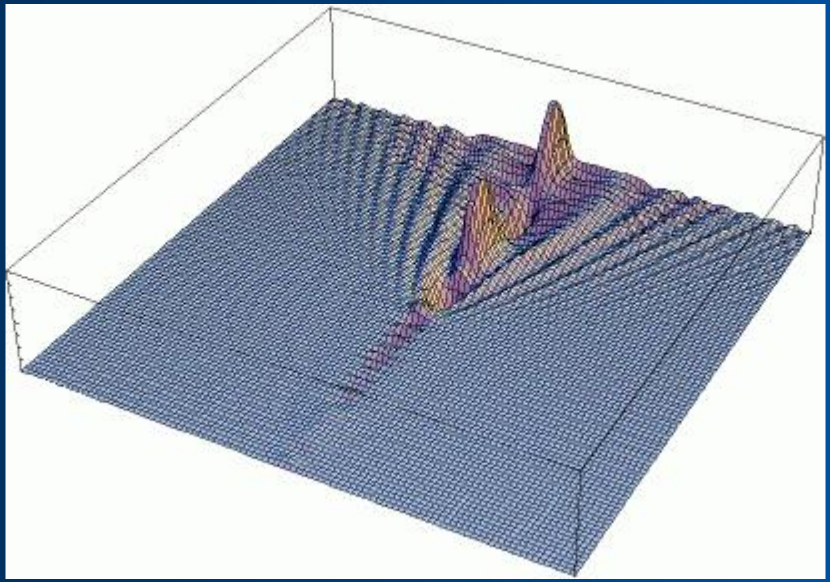
Typical shape of positive potential KdV soliton (in arbitrary units)

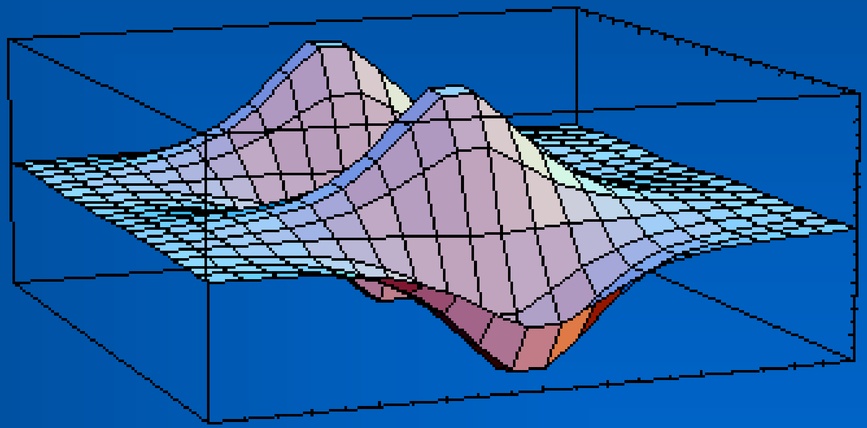
Typical interaction between two positive potential KdV solitons

Solitons nowadays



- Today solitons take apart in many physicals areas, like hydrodynamics, quantum mechanics, particle physics and so on.
- It was found a lots of equation with soliton-type solutions.





Fractional Diffusion Equations (1985, 1989)

- In many physical situations $X(t) \propto t^{\alpha}, \quad \alpha \neq 1/2$ holds (non-Fickian diffusion)
- Following scaling arguments one can postulate (and sometimes derive) fractional equations for anomalous diffusion:

$$\frac{\partial}{\partial t} P(x,t) = K \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \frac{\partial^2}{\partial x^2} P(x,t)$$

or

$$\frac{\partial}{\partial t} P(x,t) = K \frac{\partial^{2\beta}}{\partial x^{2\beta}} P(x,t)$$

Such equations allow for

- easier introduction of external forces
- introduction of boundary conditions
- using the methods of solutions known for "normal" PDEs

Reasonable values of the orders of derivatives:

$0 < \alpha < 1$ for subdiffusion; $0 < \beta < 1$ for superdiffusion.

Fractional Derivatives

1695 Leibnitz - de l'Hospital

$$\frac{d^n}{dt^n} t^m = \frac{m!}{(m-n)!} t^{m-n} \equiv \frac{\Gamma(1+m)}{\Gamma(1+m-n)} t^{m-n}$$

Trivial generalization: $$\frac{d^\nu}{dt^\nu} t^\mu = {}_0 D_t^\nu t^\mu = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\nu)} t^{\mu-\nu}$$

Interesting: $${}_0 D_t^\nu 1 = \frac{1}{\Gamma(1-\nu)} t^{-\nu}$$

This definition is enough to handle the functions which can be expanded into Taylor series, but obscures the nature of the fractional differentiation operator.

All modern definitions are based on generalizations of the repeated integration formula:

$$ {}_aD_x^{-n} f(x) = \int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} f(y_n) dy_n \ldots dy_1 = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y) dy $$

Its generalization is: The fractional integral

$$ {}_{t_0}D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_{t_0}^t \frac{f(t')}{(t-t')^{1-p}} \qquad (0 < p < 1) $$

Fractional derivatives may be defined through additional differentiation:

$$ {}_{t_0}D_t^{q} f(t) = \frac{d^n}{dt^n} {}_{t_0}D_t^{-(n-q)} f(t) \qquad (n = [q+1]) $$

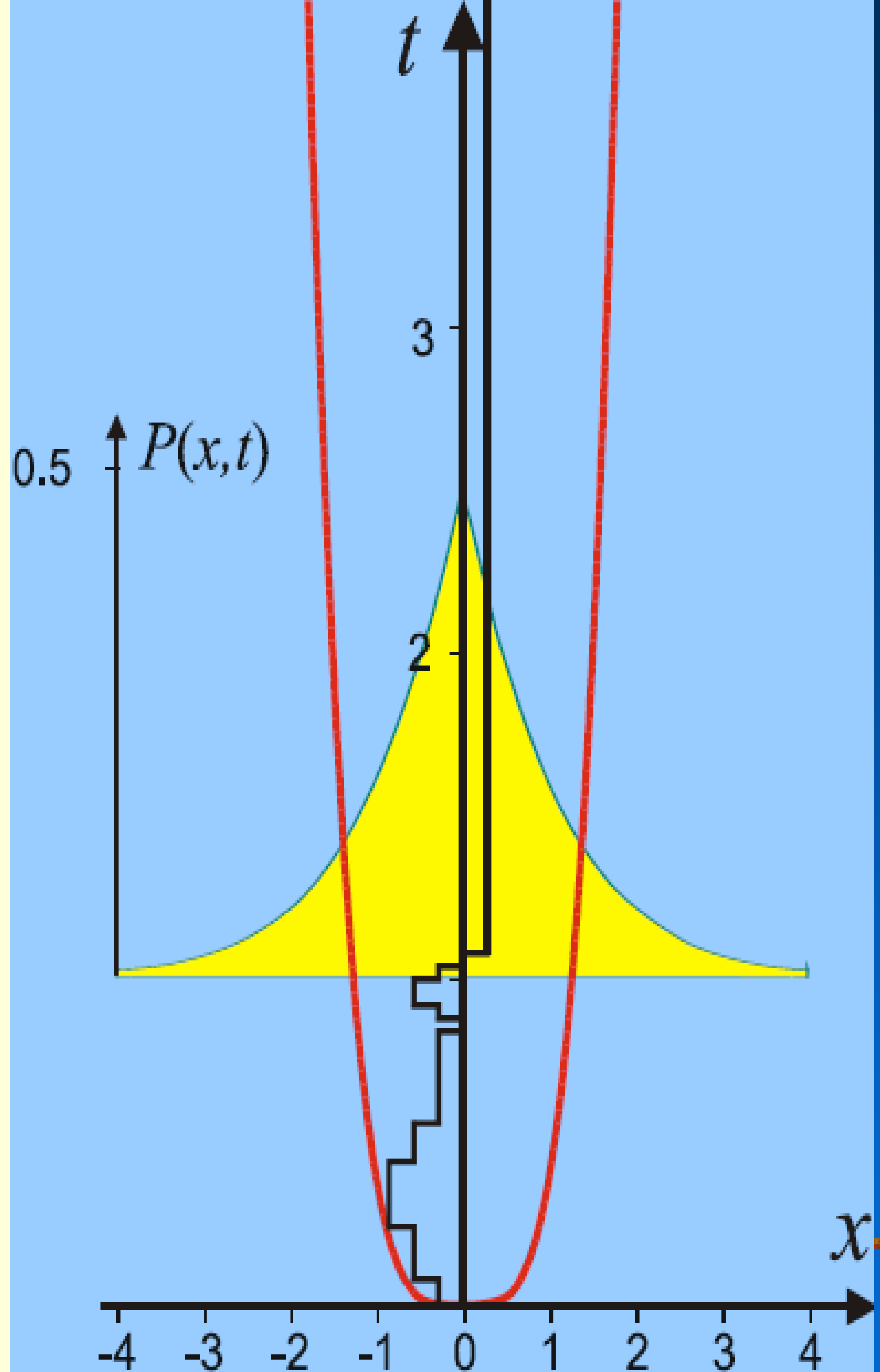
Fractional derivatives are nonlocal integral operators and are best suited for the description of nonlocalities in space (long jumps) or time (memory effects)

Examples of Fractional Calculus with $\alpha = \pm 1/2$

Semi-integral	Function	Semi-derivative
${}_0D_x^{-1/2} f(x) = \dfrac{d^{-1/2}}{dx^{-1/2}} f(x)$	$f(x)$	${}_0D_x^{1/2} f(x) = \dfrac{d^{1/2}}{dx^{1/2}} f(x)$
$2C\sqrt{x/\pi}$	C, any constant	$C/\sqrt{\pi x}$
$\sqrt{\pi}$	$1/\sqrt{x}$	0
$x\sqrt{\pi}/2$	$\sqrt{x}$	$\sqrt{\pi}/2$
$4x^{3/2}/3\sqrt{\pi}$	x	$2\sqrt{x/\pi}$
$\dfrac{\Gamma(\mu+1)}{\Gamma(\mu+3/2)} x^{\mu+1/2}$	x^{μ}, $\mu > -1$	$\dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)} x^{\mu-1/2}$
$\exp(x)\,\mathrm{erf}(\sqrt{x})$	$\exp(x)$	$1/\sqrt{\pi x} + \exp(x)\,\mathrm{erf}(\sqrt{x})$
$2\sqrt{\pi/x}\,[\ln(4x) - 2]$	$\ln x$	$\ln(4x)/\sqrt{\pi x}$

$$\frac{\partial}{\partial t}P(x,t) = {}_tD_0^{\,1-\alpha} K \frac{\partial^2}{\partial x^2} P(x,t')$$

$${}_tD_0^{\,1-\alpha} f(t) = \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-t')^{\alpha}} f(t')dt'$$



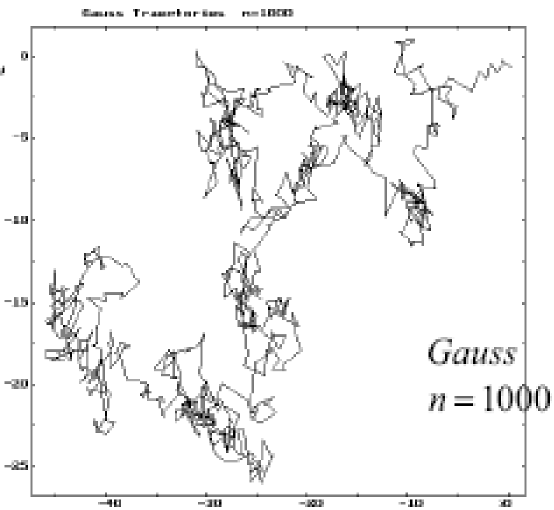
Possible positions of derivatives:

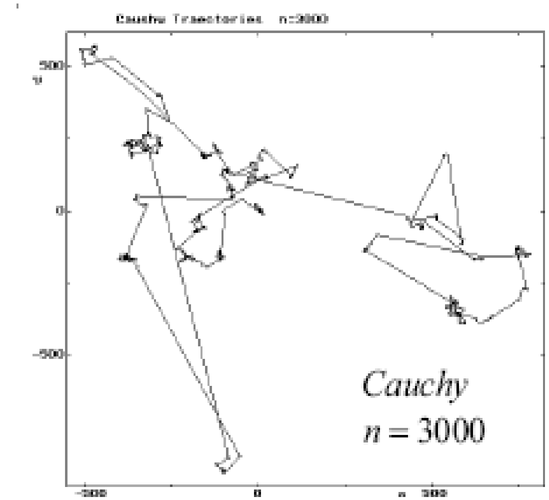
Normal forms:	Modified forms:
$$\frac{\partial^\alpha}{\partial t^\alpha} P(x,t) = K \frac{\partial^2}{\partial x^2} P(x,t)$$	$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} K \frac{\partial^2}{\partial x^2} P(x,t)$$
Caputo derivative on the “correct” side (l.h.s.)	Riemann-Liouville derivative on the “wrong” side (r.h.s.)
$$\frac{\partial}{\partial t} P(x,t) = K \frac{\partial^{2\beta}}{\partial x^{2\beta}} P(x,t)$$	$$-\frac{\partial^{2-2\beta}}{\partial x^{2-2\beta}} \frac{\partial}{\partial t} P(x,t) = K \frac{\partial^2}{\partial x^2} P(x,t)$$
Riesz-Weyl derivative on the “correct” side (r.h.s.)	Riesz-Weyl derivative on the “wrong” side (l.h.s.)

Normal vs Anomalous Diffusion

Normal diffusion $\left\langle \left(\vec{R}-\vec{R}_0\right)^2 \right\rangle = c_{\dim} D t \propto t^1$

Bachelier (1900), *Einstein* (1905)





Anomalous diffusion

$\left\langle \left(\vec{R}-\vec{R}_0\right)^2 \right\rangle \propto t^\mu, \ \mu \neq 1$

Superdiffusion

$\mu > 1$

- *turbulent media (gases, fluids, plasmas)*
- *Hamiltonian chaotic systems*
- *deterministic maps*
- *foraging movement*

Subdiffusion

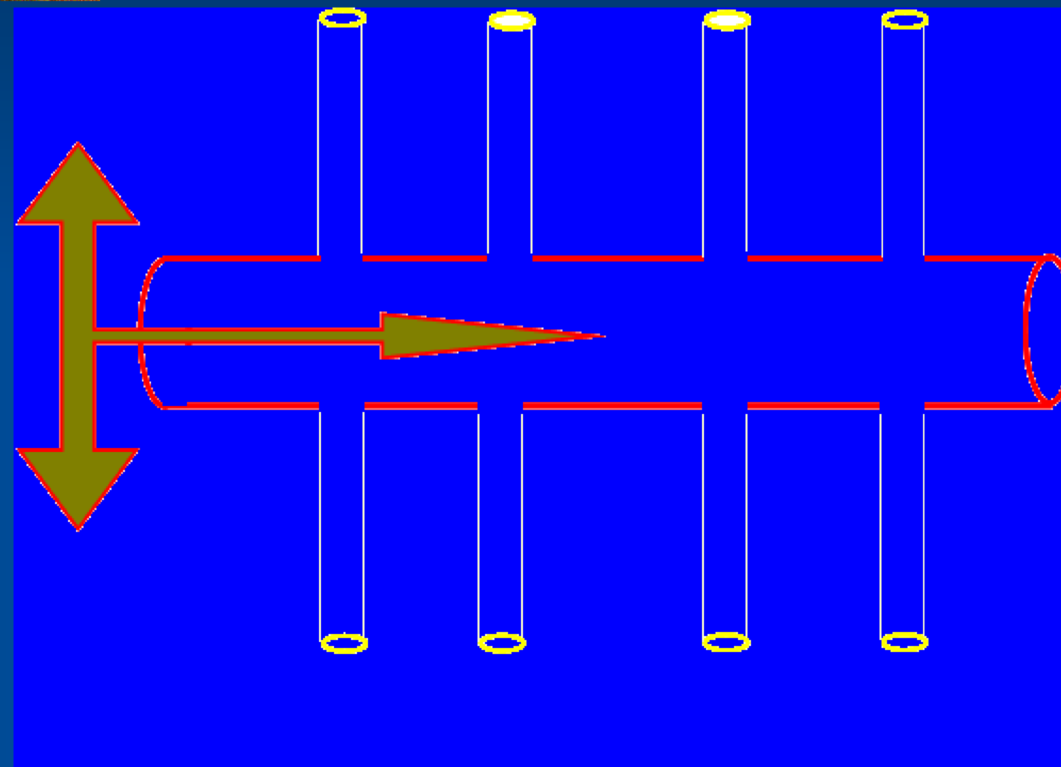
$\mu < 1$

- *turbulent plasmas*
- *transport on fractals*
- *contamimants in underground water*
- *amorphous solids*
- *convective patterns*
- *polymeric systems*
- *deterministic maps*

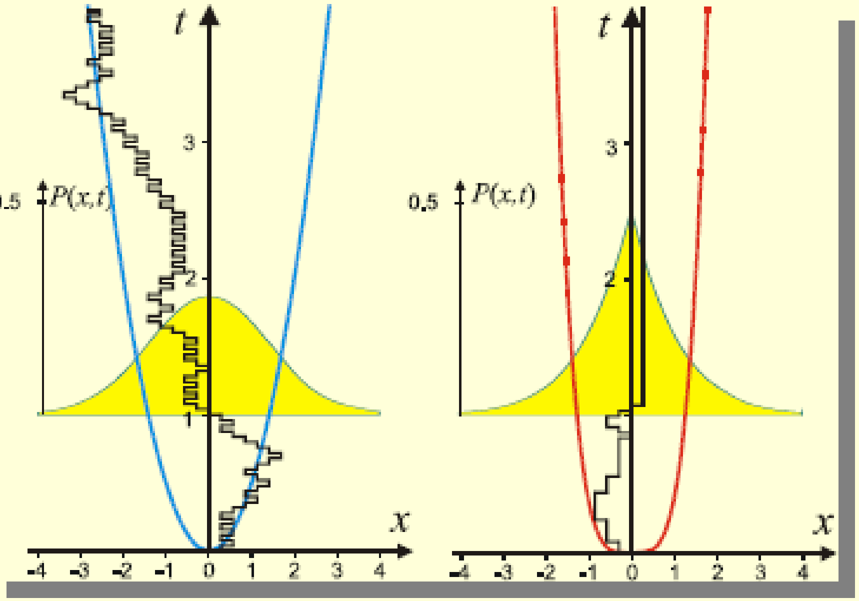
Comb-like model

$$\frac{\partial}{\partial t}P(x,t) = K\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\frac{\partial^2}{\partial x^2}P(x,t)$$

$$\frac{\partial}{\partial t}P(x,t) = K\frac{\partial^{2\beta}}{\partial x^{2\beta}}P(x,t)$$



Some Solutions

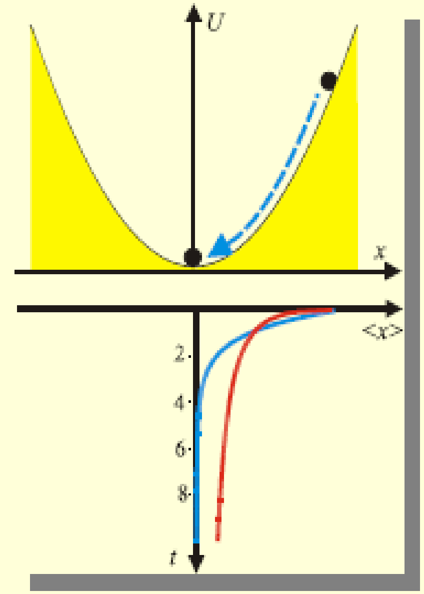


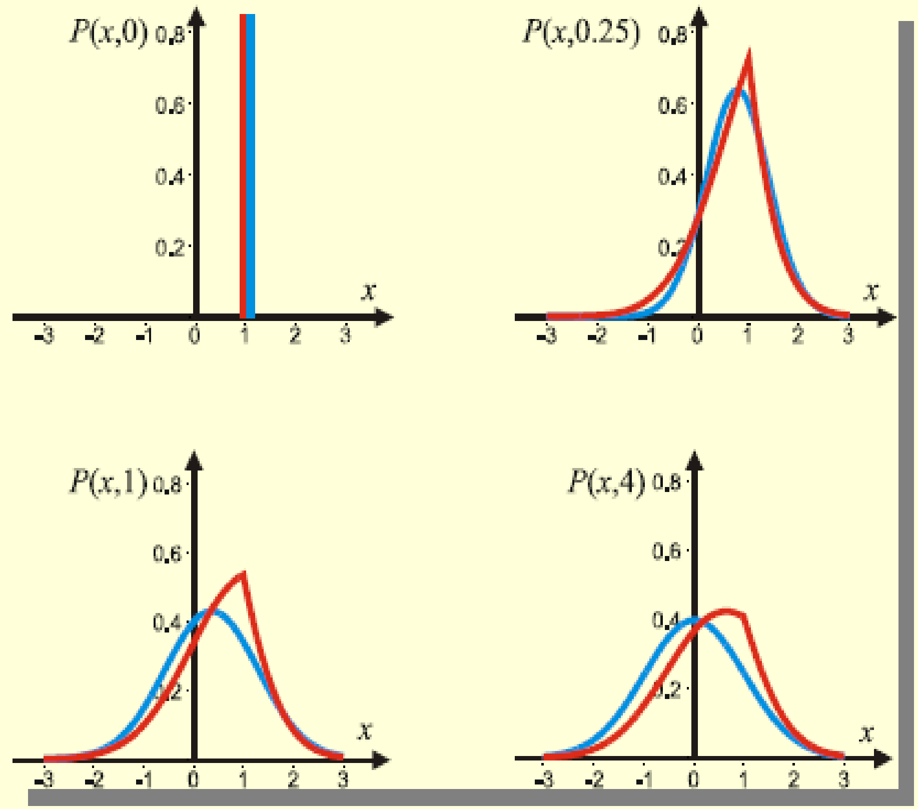
Free diffusion:

Left: Normal diffusion

Right: Subdiffusion with $\alpha = 1/2$

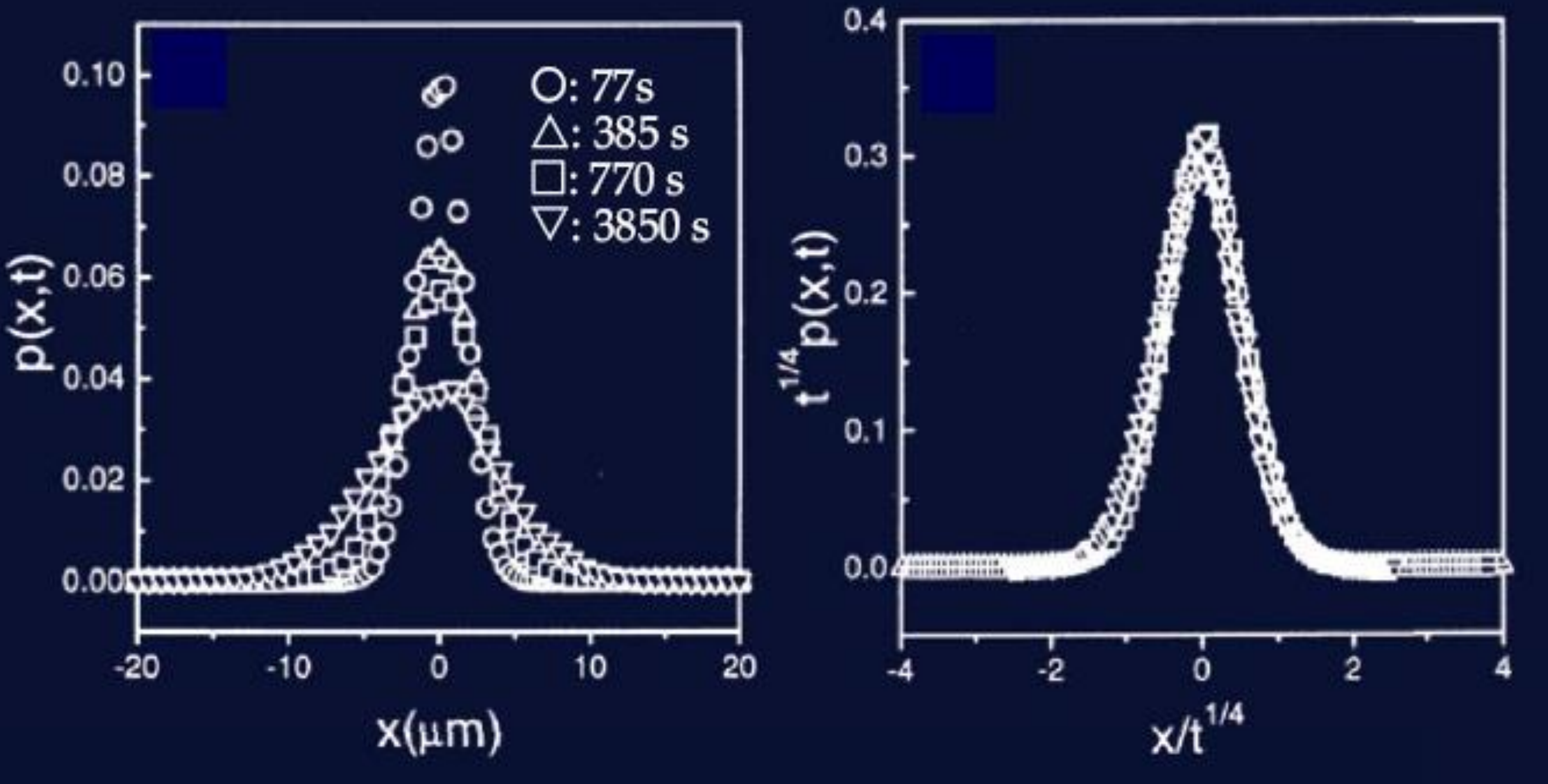
Ornstein-Uhlenbeck process: Diffusion in a harmonic potential





Propagator

$$p(x,t)_{x=0,t=0} = \frac{1}{\sqrt{4\pi F t^{1/4}}} \exp\left(-x^2/4Ft^{1/2}\right) \qquad \text{(hard rods)}$$



The Fractional Fokker-Planck Equation

$$\dot{P}(x,t) = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\frac{V'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2}\right]P(x,t)$$

$$[\eta_\alpha] = \sec^{\alpha-2}$$

- Force-free mean squared displacement

$$< x^2(t) >_0 = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha$$

- Stationary solution

$$P_{st} \propto \exp\left(-\frac{V(x)}{k_B T}\right)$$

$$K_\alpha = \frac{k_B T}{m\eta_\alpha}$$ **Generalized Einstein-Stokes relation**

Thank you!